LUNEDÌ 8.30-10.30 TEORIA
MERCOLEDÌ 14.30 - 16.30 ESERCIZI
GIOVEDÌ AULA E ESERCIZI INDIVIDUALI

MARINO.BELLONI@UNIPR.IT

WWW2.UNIPR.IT/BELMAR68/

ESAMI: nessun compitino

Esame scritto: due prove:

- 8 quiz a 4 risposte $+3$ giusto -1 sbagliato. Se voto ≥ 10 (max 24)
- 4 esercizi

$\frac{\text{Voto } 1}{2} + \text{Voto } 2$ Esame orale facoltativo.

TESTO: "Calcolo differenziale e integrale per funzioni di più variabili" M. BELLONI e L. LORENZI, casa editrice PITAGORA.

PROGRAMMA

- Curve
- Funzioni di più variabili
- ODE (equazioni differenziali)
- Integrazione doppia (e tripla)

$\varphi: I \to \mathbb{R}^2$

$I \subseteq \mathbb{R}$ intervallo

$\varphi(t) = (\varphi_1(t), \varphi_2(t))$ è continua (ovvero $\varphi_1: I \to \mathbb{R}$ continua e $\varphi_2: I \to \mathbb{R}$ continua).

Vettore che ad ogni t genera un vettore. Questa è una CURVA PIANA (funzione che va da un intervallo a $\mathbb{R}^2$ continua).

- $\varphi(I) \subset \mathbb{R}^2$ si dice SOSTEGNO DELLA CURVA (immagine)
- φ si dice CHIUSA se $I = [a, b]$ e $\varphi(a) = \varphi(b)$ (il cerchio)
- φ si dice SEMPLICE se $\varphi(t_1) \neq \varphi(t_2)$ $\forall t_1, t_2 \in I$, $t_1 \neq t_2$, $t_1, t_2 \in$ inter di I.

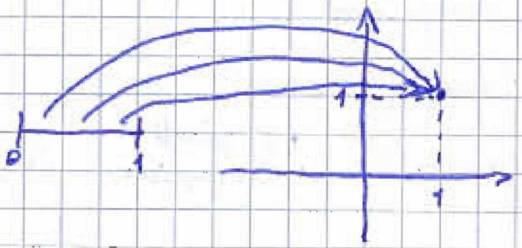
(senza nodi, iniettiva, almeno una componente iniettiva).

- φ è DERIVABILE in $t_0 \in I$ se φ_1 e φ_2 sono derivabili in t_0.
- φ è di CLASSE C^1 se φ_1 e φ_2 di classe C^1.

- φ è REGOLARE se $\varphi'(t) \neq (0,0)$ $\forall t \in I$
- φ è REGOLARE A TRATTI se $\varphi'(t) \neq (0,0)$ $\forall t \in I \setminus \{t_1, t_2, \dots, t_k\}$, cioè tranne in un numero finito di punti.

ESEMPI

① $\varphi(t) = (1,1)$ $\forall t \in [0,1]$



$\varphi_1(t) = 1$ è continua

$\varphi_2(t) = 1$ è continua sostegno $\varphi([0,1]) = \{(1,1)\}$

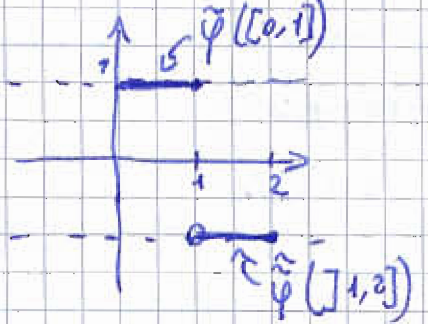
La curva φ è chiusa, non è semplice, è derivabile, di classe C^1

$\varphi_1(t)' = 0$
$\varphi_2(t)' = 0$ $\forall t \in [0,1]$ φ non è regolare perché $\varphi'(t) = 0$ $\forall t \in [0,1]$

②
$$\varphi(t) = \begin{cases} (t,1) & t \in [0,1] \\ (t,-1) & t \in]1,2] \end{cases}$$
non è una curva

$\tilde{\varphi}(t) = (t,1)$ $t \in [0,1]$

$\tilde{\tilde{\varphi}}(t) = (t,-1)$ $t \in [1,2]$



La seconda componente $\varphi_2(t)$ non è continua, perché uguale a 1 se $t \in [0,1]$ e -1 se $t \in]1,2]$

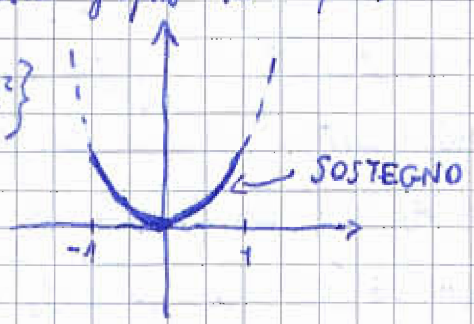
$\Rightarrow \varphi$ non è continua $\Rightarrow \varphi$ non è una curva.

③
$\varphi(t) = (t, t^2)$ $t \in [-1,1]$

$\varphi_1(t) = t$ continua
$\varphi_2(t) = t^2$ continua $[-1,1]$ è un intervallo $\Rightarrow \varphi(t)$ è una curva

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$$
$y = x^2$ La curva giace sul grafico di $y = x^2$

$\varphi([-1,1]) = \{(x,y) : -1 \leq x \leq 1,\ y = x^2\}$



$\varphi(-1)=(-1,1)$ $\varphi(1)=(1,1)$ La curva non è chiusa

$\varphi(t_1)\neq\varphi(t_2)$ $\forall t_1\neq t_2 \in]-1,1[$ poiché $\varphi_1(t_1)=t_1\neq t_2=\varphi_1(t_2)$. La curva è semplice.

FUNZIONI DI CLASSE C^1 → funzioni la cui derivata prima è continua

$\varphi'(t)=(1,2t)$

φ è derivabile, di classe C^1 e regolare

④ $\varphi(t)=(t^2-1,\ t^3-t)$ $t\in[-2,2]$

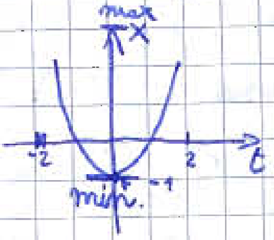
$\varphi_1(t)=t^2-1$
$\varphi_2(t)=t^3-t$ sono continue, derivabili con derivata continua

$[-2,2]$ è un intervallo
⇒ φ è una curva derivabile di classe C^1

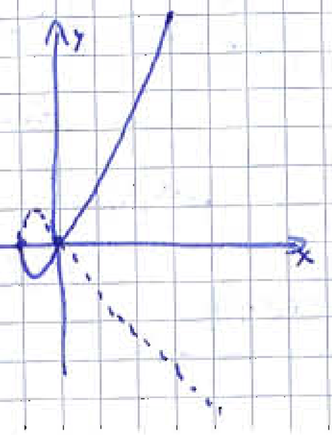
$\varphi'(t)=(2t,\ 3t^2-1)$ Voglio provare φ regolare:
cerco per quali t si ha $\varphi'(t)=(0,0)$ $\begin{cases}2t=0\\3t^2-1=0\end{cases}$ $\begin{cases}t=0\\-1=0\end{cases}$ MAI
$\varphi'(t)\neq(0,0)$ $\forall t$
⇒ φ è regolare

$x(t)=t^2-1$
$y(t)=t^3-t=t(t^2-1)$ $\begin{cases}t^2=x+1\\y(t)=t\cdot x(t)\end{cases}$ $\begin{cases}t=\sqrt{x+1}\\y=\sqrt{x+1}\cdot x\end{cases}$ e $\begin{cases}t=-\sqrt{x+1}\\y=-\sqrt{x+1}\cdot x\end{cases}$

Se $t\in[-2,2]$, $x=t^2-1$ va da $-1\leq x\leq 3$
valore minimo di t^2 è 0.



$y=x\sqrt{x+1}$ $x\in[-1,3]$ —
sono speculari
$y=-x\sqrt{x+1}$ $x\in[-1,3]$ ...



Il sostegno è l'unione delle curve
La curva non è semplice perché in
$x=0$ c'è un nodo, cioè $t^2-1=0$ quando $t=\pm1$

$\varphi(1)=\varphi(-1)=(0,0)$ $\varphi(0)=(-1,0)$ $\varphi(-2)=(3,-6)$

La retta tangente nel punto $x=0$ e $y=0$ non esiste

(come $y=|x|$ )

φ non è chiusa.

⑤

$\varphi(t)=(\cos t, \operatorname{sen} t)$ $t\in[0,2\pi]$ circonferenza di raggio 1 con centro in(

$\begin{cases}\varphi_1(t)=\cos t\\ \varphi_2(t)=\operatorname{sen} t\end{cases}$ sono continue, derivabili; con derivata continua

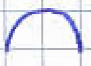
$I\in[0,2\pi] \Rightarrow \varphi$ è una curva derivabile di classe C^1.

Regolare?

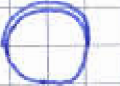
$\varphi'(t)=(-\operatorname{sen} t, \cos t)$ $\begin{cases}-\operatorname{sen} t=0\\ \cos t=0\end{cases}$ non ha soluzioni $\Rightarrow \varphi$ è regolare

$\begin{cases}x(t)=\cos t\\ y(t)=\operatorname{sen} t\end{cases}$ $x^2+y^2=\cos^2 t+\operatorname{sen}^2 t=1$ I punti $\varphi(t)$ giacciono sulla cfr $x^2+y^2=1$

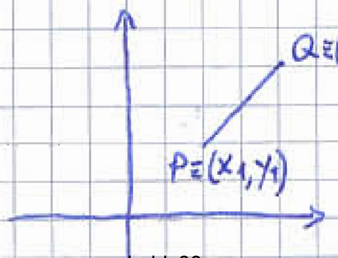
- Il sostegno è $\varphi(I)=\{(x,y): x^2+y^2=1\}$
- $\varphi(0)=\varphi(2\pi)=(0,0) \Rightarrow \varphi$ è chiusa
- φ è semplice $\begin{cases}\cos t_1=\cos t_2\\ \operatorname{sen} t_1=\operatorname{sen} t_2\end{cases} \Leftrightarrow \varphi(t_1)=\varphi(t_2) \Leftrightarrow t_1=t_2 \Leftrightarrow \varphi$ è sempli

Se $t\in[0,\pi]$, le proprietà sarebbero tutte valide tranne il fatto che φ è chiusa 

Se $t\in[0,3\pi]$, l'unica proprietà che non vale è la semplicità e

 $\varphi\left(\frac{\pi}{5}\right)=\varphi\left(\frac{\pi}{6}+2\pi\right)$ il fatto che è chiusa $\varphi(0)\neq\varphi(3\pi)$

EQUAZIONE DI SEGMENTO DI RETTA



$\varphi(t)=tP+(1-t)Q$ $t\in[0,1]$

$=t(x_1,y_1)+(1-t)(x_2,y_2)=tx_1+(1-t)x_2,\ ty_1+(1-t)y_2=$

$=\left(x_2+t(x_1-x_2);\ y_2+t(y_1-y_2)\right)$ $\varphi(0)=Q$ punto iniziale

$\varphi(1)=P$ punto finale

$\gamma(t)=tQ+(1-t)P \ \ldots = \left(x_1+t(x_2-x_1);\ y_1+t(y_2-y_1)\right)$ con $t\in[0,1]$

$\gamma(0)=P$ punto iniziale

$\gamma(1)=Q$ punto finale

Il segmento di retta

- è una curva
- ha come sostegno il segmento PQ $\left\{(x,y):\ x_1\le x\le x_2 \text{ e } y=x\frac{y_2-y_1}{x_2-x_1}+\frac{y_1x_2-x_1}{x_2-x_1}\right.$

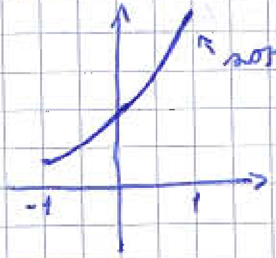
La retta per P e Q ha equazione

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \qquad y-y_1=(x-x_1)\cdot\frac{y_2-y_1}{x_2-x_1} \ \ldots \ y=x\frac{y_2-y_1}{x_2-x_1}+\frac{y_1x_2-x_1y_2}{x_2-x_1}$$

- è semplice

POSSO TROVARE UNA CURVA CHE STIA
SU UN SOSTEGNO DATO

$y=e^x+1 \quad x\in[-1,1] \equiv S$



$\varphi(t)=(t;\ e^t+1)$ con $t\in[-1,1]$

$\varphi([-1,1])=S=$ sostegno di φ.

$y=x^3+2\log x \quad x\in[2,3] \neq \varphi(t)=(t;\ t^3+2\log t)$ che ha come

$S=\{(x,y):\ 2\le x\le 3,\ y=x^3+2\log x\}$ sostegno S con $t\in[2,3]$

$\uparrow$ FUNZIONE $\qquad\qquad$ $\uparrow$ SOTTOINSIEME DI $\mathbb{R}^2$

insieme di punti $\qquad$ applicazione che ha come immagine l'insieme di punti

Potevo anche prendere

$\varphi(t)=\left(3t;\ (3t)^3+2\log(3t)\right)$ $t\in\left[\frac{2}{3},1\right]$ percorrerò il sostegno più veloceme

$\varphi(t)=\left(\frac{t}{4};\ \left(\frac{t}{4}\right)^3+2\log\left(\frac{t}{4}\right)\right)$ $t\in[8,12]$ percorso il sostegno più lentamen

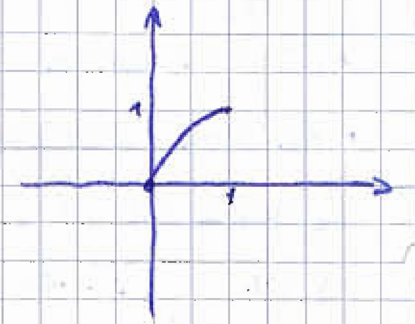
DEF.

$\varphi : I \to \mathbb{R}^2$ e $\psi : J \to \mathbb{R}^2$ due curve tali che $\varphi(I) = \psi(J) = S$ queste si dicono "equivalenti" se esistono $g : I \to J$ biettiva di classe C^1 $g'(t) \neq 0$ $\forall t \in I$ tale che $\psi(g(t)) = \varphi(t)$

ESEMPIO

$$\varphi(t) = \begin{cases} (1-t, 2t-t^2) & t \in [0,1] \\ (1-t, 1) & t \in]1,2] \end{cases}$$

$t \in [0,1]$ $\begin{cases} x(t) = 1-t \\ y(t) = 2t\left(1-\frac{t}{2}\right) \end{cases}$ $0 \leq 1-t \leq 1 \to 0 \leq x \leq 1$ $\quad y = 2\left(1-\frac{x}{2}\right) \cdot x = 2x - x^2$

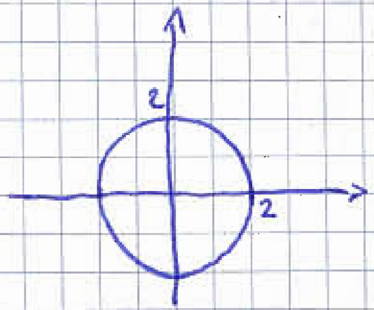


$S = \{(x,y) : 0 \leq x \leq 1,\ y = 2x - x^2\}$

$t \in]1,2]$ $\begin{cases} x(t) = 1-t \\ y(t) = 1 \end{cases}$ $\begin{cases} t = 1-x \\ y = 1 \end{cases}$ $- \leq 1-t \leq 0$

ESEMPIO

$\varphi(t) = (2\cos t, 2\,\mathrm{sen}\,t)$ $t \in [0, 2\pi] = I$ $\begin{cases} x(t) = 2\cos t \\ y(t) = 2\,\mathrm{sen}\,t \end{cases}$ $x^2 + y^2 = 4$



$S = \varphi(I) = \{(x,y) : x^2 + y^2 = 4\}$

$\psi(t) = (2\cos t\pi, 2\,\mathrm{sen}\,\pi t)$ $t \in [0,2] = J$ $x^2 + y^2 = 4$

$g(t) = \frac{t}{\pi}$ $t \in [0, 2\pi]$ $g : [0, 2\pi] \to [0,2]$ biettiva di classe C^1

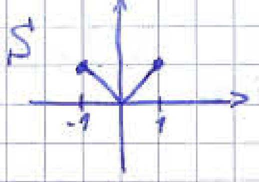
$g'(t) = \frac{1}{\pi} \neq 0.$

$\varphi(t) = (t, |t|)$ $t \in [-1,1]$ $\varphi : \begin{cases} x(t) = t \\ y(t) = |t| \end{cases}$ $t \in [-1,1]$ $\varphi'(0) \nexists$ perchè $y'(t) = \frac{t}{|t|} \ldots$

$\Rightarrow \varphi$ è regolare a tratti.

φ è semplice, non chiusa.

$\begin{cases} x(t) = t \\ y(t) = |t| \end{cases}$ $-1 \leq t \leq 1$ $S = \{(x,y) : y = |x|, -1 \leq x \leq 1\} = \varphi(I)$



$\psi(t) : \begin{cases} x(t) = t^3 \\ y(t) = |t|^3 \end{cases}$ ψ è una curva

$\exists \psi'(0) : \psi'(t) = \left(3t^2 ; 3|t|^2 \cdot \frac{t}{|t|}\right) = \left(3t^2 ; 3t|t|\right)$

ψ' è continua $\Rightarrow$ ψ è di classe C^1 $\qquad \psi(J) = \varphi(I)$

$\psi(0) = (0,0) \Rightarrow \psi$ non è regolare, ma regolare a tratti

ψ è semplice, non chiusa.

Le due figure sono equivalenti?

$g(t) = t^3$ $\quad g : I \to J$ biettiva di classe C^1, ma $g'(0) = 0$

$\Rightarrow$ non sono equivalenti.

DEF

Data $\varphi : I \to \mathbb{R}^2$ curva, se $\exists \varphi'(t_0)$, $t_0 \in I$, diciamo $\varphi'(t_0) \equiv$ vettore tangente a φ nel punto $\varphi(t_0)$.

Se $\varphi(t) \neq \varphi(t_0)$ $\forall t \neq t_0$, allora la retta $(x,y) = \varphi(t_0) + \lambda \varphi'(t_0)$ $\lambda \in \mathbb{R}$ è detta "RETTA TANGENTE a φ nel punto $\varphi(t_0)$".

Cioè la curva deve passare per $\varphi(t_0)$ una sola volta (curva semplice in quel punto).

ESEMPIO

$y = 3x^2 = f(x)$ $-1 \leq x \leq 1$ $f'(x)$ $y' = 6x$ La retta tangente è $y = f(1) + f'(1)(x-1)$

$\to y = 3 + 6(x-1) = 3 + 6x - 6 = -3 + 6x$

La curva che parametrizza questo sostegno è: $\begin{cases} x(t) = t \\ y(t) = -3 + 6t \end{cases}$

$\varphi \begin{cases} x(t) = t \\ y(t) = 3t^2 \end{cases}$ $-1 \leq t \leq 1$ $\quad \varphi([-1,1]) = \{(x,y) : x \in [-1,1], y = 3x^2\}$

parametrizzazione di $f(x)$

$\varphi'(t) = (1, 6t)$

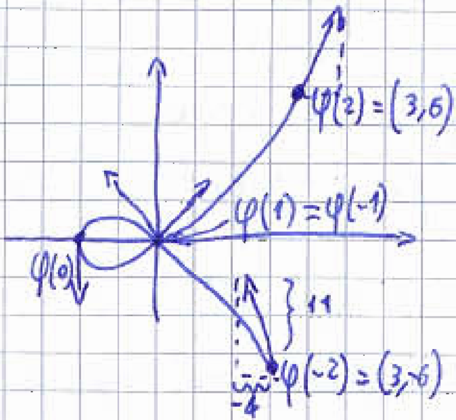
$(1,3) = (t, 3t^2)$ $\quad$ Considero $\varphi(1)$ e $\varphi'(1)$ $\quad \varphi(1) = (1,3)$

$\hookrightarrow t = 1$ $\qquad \varphi'(1) = (1,6)$

RETTA TANGENTE $(x,y) = (1,3) + \lambda(1,6)$ $\begin{cases} x = 1+\lambda \\ y = 3+6\lambda \end{cases}$ $\lambda \in \mathbb{R}$ che ha come sostegno la retta tangente calcolata nel vecchio modo

ESEMPIO 2

$\varphi(t) = (t^2-1, t^3-t)$ $t \in [-2,2]$



$\varphi'(t) = (2t, 3t^2-1)$

$\varphi(-2) = (3,-6)$ $\varphi(-1) = (0,0)$ $\varphi(0) = (-1,0)$ $\varphi(1) = (0,0)$ $\varphi(2) = (3,6)$

$\varphi'(-2) = (-4, 11)$ $\varphi'(-1) = (-2, 2)$ $\varphi'(0) = (0,-1)$ $\varphi'(1) = (2,2)$ $\varphi'(2) = (4,11)$

retta tangente in -2 $(x,y) = \varphi(-2) + \lambda\varphi'(-2)$ $\begin{cases} x = 3+\lambda(-4) \\ y = -6+\lambda(11) \end{cases}$ $\begin{cases} x = 3-4\lambda \\ y = -6+11\lambda \end{cases}$

↗ equazione parametrica

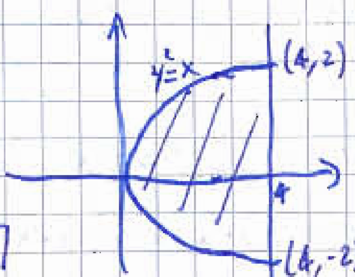
$$\begin{cases} \lambda = \dfrac{3-x}{4} \\ y = -6 + \dfrac{11}{4}(3-x) \end{cases} \quad 4y = -24+33-11x \quad 4y+11x = 9$$ ⇐ equazione cartesiana

Il vettore tangente esiste sempre, la tangente no (come in 0,0).

$S = \{(1,1), (2,1)\}$ . . NON esiste una curva che abbia questo sostegno perchè non è un intervallo.

$A = \{(x,y) : y^2 \leq x \leq 4\}$ S bordo di A

$\exists\, \varphi : I \to \mathbb{R}^2$ t.c. $\varphi(I) = S$? Sì



VETTORE TANGENTE ⇒ direzione della retta tangente

TANGENTE = PUNTO + λ · VETTORE TANGENTE.

$S \equiv$ bordo di $A = \overset{①}{\{(x,y) : -2 \leq y \leq 2,\ x = y^2\}} \cup \overset{②}{\{(x,y) : x = 4, -2 \leq x \leq 2\}}$ parametrizzazi

① $\varphi: \begin{cases} x = t^2 \\ y = t \end{cases} \quad t \in [-2,2]$

② $\varphi: \begin{cases} x = 4 \\ y = -t \end{cases} \quad t \in [-2,2]$

$\varphi'(t) = (2t, 1) \quad \varphi'(0) = (0,1)$

mi muovo in senso orario, quindi in ② adeguo la y con un −.

Devo unire ① e ②

Prendo λ in modo che vari tra $2 \leq \lambda \leq 6$ perchè il range deve essere = a quello $[-2,2]$, cioè 4, partendo da 2

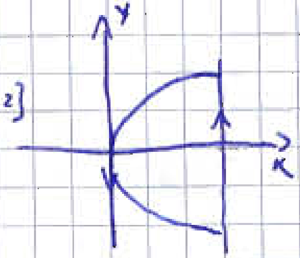
$$\begin{cases} x = 4 \\ y = -t + 4 \end{cases} \quad t \in [2,6]$$

$$\phi(t) = \begin{cases} (t^2, t) & -2 \leq t \leq 2 \\ (4, -t+4) & 2 < t \leq 6 \end{cases}$$ altre param: $$\phi(t) = \begin{cases} (4t^2, 2t) & -1 \leq t \leq 1 \\ (4, -2t+4) & 1 < t \leq 3 \end{cases}$$

Cercare la parametrizzazione per il senso antiorario

② $\begin{cases} x = 4 \\ y = t \end{cases} \quad t \in [-2,2]$

① $\begin{cases} x = t^2 \\ y = t \end{cases} \quad \varphi'(0) = (0,-1) \quad t \in [-2,2]$



$$\begin{cases} x = 4 \\ y = t - 4 \end{cases}$$

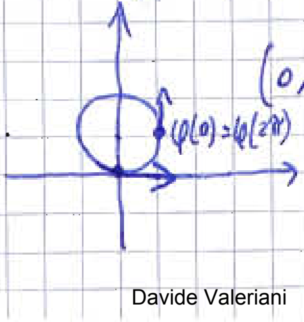
$$\phi(t) = \begin{cases} (t^2, -t) & \text{se } t \in [-2,2] \\ (4, t-4) & \text{se } t \in [2,6] \end{cases}$$

$$\begin{cases} x = \cos t \\ y = \operatorname{sen} t + 1 \end{cases} \quad t \in [0, 2\pi] \quad \text{circonferenza}$$

$$\begin{cases} x = \cos t \\ y - 1 = \operatorname{sen} t \end{cases}$$ $x^2 + (y-1)^2 = 1$ con questa parametrizzazione mi muov

in senso antiorario. Infatti

$\varphi' = (-\operatorname{sen} t, \cos t)$



$(0,0) = \varphi(t) \iff t = \frac{3}{2}\pi$

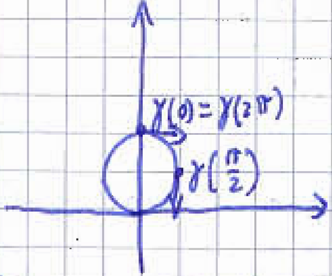
$\varphi'(\frac{3}{2}\pi) = (1,0)$

$\varphi'(0) = (0,1)$

$$\gamma: \begin{cases} x = \operatorname{sen} t \\ y = 1 + \cos t \end{cases} \quad t \in [0, 2\pi] \qquad x^2 + (y-1)^2 = 1$$

$\gamma(0) = (0, 2)$

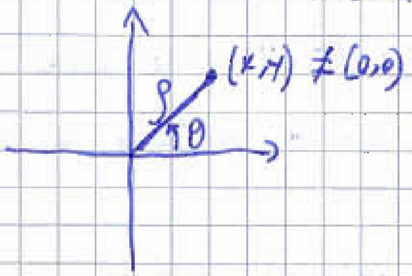
$\gamma\left(\frac{\pi}{2}\right) = (1, 1)$



SENSO ORARIO

$\gamma'(t) = (\cos t, -\operatorname{sen} t) \qquad \gamma'(0) = (1, 0)$

COORDINATE POLARI



$\rho = \sqrt{x^2 + y^2}$ MODULO

$$\theta = \begin{cases} \operatorname{arctg} \frac{y}{x} & \text{se } x > 0 \\ \pi + \operatorname{arctg} \frac{y}{x} & \text{se } x < 0 \\ \frac{\pi}{2} & \text{se } x = 0 \text{ e } y > 0 \\ \frac{3}{2}\pi & \text{se } x = 0 \text{ e } y < 0 \end{cases}$$ ARGOMENTO

ESEMPIO

$(1,1)$ $\rho = \sqrt{2}$ $\theta = \operatorname{arctg} 1 = \frac{\pi}{4}$

$(0,1)$ $\rho = 1$ $\theta = \frac{\pi}{2}$

$(-1,-1)$ $\rho = \sqrt{2}$ $\theta = \pi + \operatorname{arctg} \frac{-1}{-1} = \frac{5}{4}\pi$

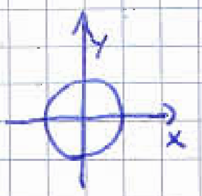
$(0,-1)$ $\rho = 1$ $\theta = \frac{3}{2}\pi$

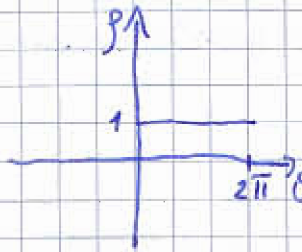
ESPRESSIONE DI CURVE IN COORDINATE POLARI

$$\Phi :]0, +\infty[\times [0, 2\pi[\longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad \Phi(\rho, \theta) = (\rho\cos\theta, \rho\operatorname{sen}\theta)$$

$x^2 + y^2 = 1$





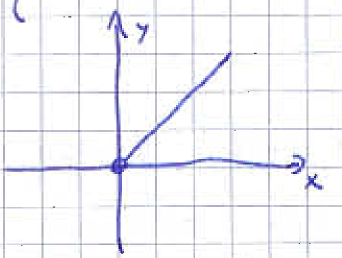
θ non ha limiti

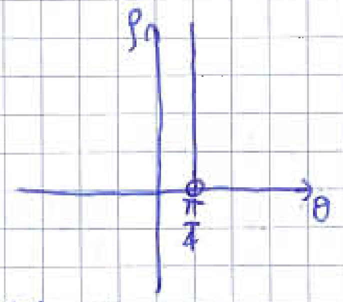
$\rho^2 = 1 \qquad \rho = +1$

~~$\rho = -1$~~ NON VA BENE perch

ρ essendo una distanza è sempre $\rho > 0$.

$\left\{(x,y) : x>0 , y=x\right\}$





$y=x \quad \rho \,sen\theta = \rho \cos\theta$

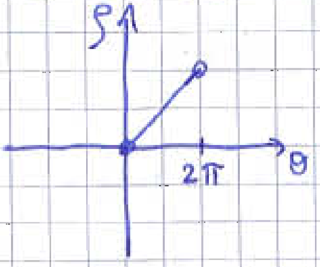
$sen\theta = \cos\theta$

$tg\,\theta = 1$

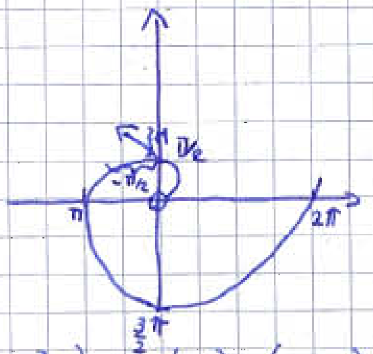
$\theta = arctg\, 1 = \frac{\pi}{4}$ $\quad \rho$ è libera, ma >0.

ESEMPIO

$\rho = \theta \quad 0<\theta<2\pi$



$$\varphi : \begin{cases} x = \rho\cos\theta = \theta\cos\theta \\ y = \rho\, sen\theta = \theta\, sen\theta \end{cases}$$



$\varphi\left(\frac{\pi}{2}\right) = \left(0, \frac{\pi}{2}\right)$

$\varphi(\pi) = (-\pi, 0) \quad \varphi\left(\frac{3}{2}\pi\right) = \left(0, -\frac{3}{2}\pi\right) \quad \varphi(2\pi) = (2\pi, 0)$

$\varphi'(\theta) = (\cos\theta - \theta\, sen\theta, \; sen\theta + \theta\cos\theta) \quad \varphi(\theta) = (\theta\cos\theta, \theta\, sen\theta)$

$\varphi'\left(\frac{\pi}{2}\right) = \left(-\frac{\pi}{2}, 1\right)$

$\varphi(\theta)\cdot\varphi'(\theta) = ? \quad \varphi(\theta)\cdot\varphi'(\theta) = \theta\cos\theta(\cos\theta - \theta\, sen\theta) + \theta\, sen\theta(\cos\theta - \theta\, sen\theta) +$

$+\theta\cos\theta(sen\theta + \theta\cos\theta) + \theta\, sen\theta(sen\theta + \theta\cos\theta) =$

$= \theta\cos^2\theta \cancel{- \theta^2 sen\theta\cos\theta} + \theta\, sen\theta\cos\theta \cancel{- \theta^2 sen^2\theta} + \theta\, sen\theta\cos\theta + \theta\cos^2\theta \cancel{+ \theta\, sen^2\theta} \cancel{+ \theta^2 sen\theta\cos\theta}$

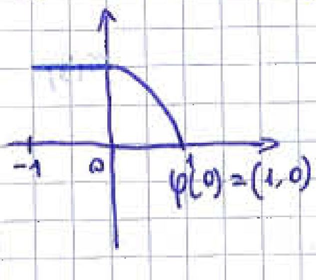
$= 2\theta\cos^2\theta + 2\theta\, sen\theta\cos\theta = 2\theta\cos\theta(\cos\theta + sen\theta)$

10/03/08

$$\varphi(t) = \begin{cases} (1-t, 2t-t^2) & t\in[0,1] \\ (1-t, 1) & t\in]1,2] \end{cases}$$

$t\in[0,1] \quad \begin{cases} x = 1-t \\ y = 2t-t^2 = 2t-t^2+1-1 = 1-(t^2-2t+1) = 1-(1-t)^2 = 1-x^2 \end{cases}$

$0\le x\le 1$
$\hookrightarrow 1-t$



$S = \left\{(x,y) : x\in[0,1] \;\; y=1-x^2\right\} \cup$

$\left\{(x,y) : x\in[-1,0[\;\; y=1\right\}$

$t\in]1,2] \quad \begin{cases} x = 1-t \\ y = 1 \end{cases} \qquad -1\le 1-t<0$

$\Downarrow$

φ è semplice perchè la prima componente è sempre uguale a $1-t$, che è iniettiva.

φ non è chiusa perchè $\varphi(0) \neq \varphi(2)$

φ è di classe C^1 $\forall t \in [0,2] \setminus \{1\}$ $\qquad \varphi'(t) = \begin{cases} (-1, 2-2t) & \forall t \in [0,1] \\ (-1, 0) & \forall t \in]1,2] \end{cases}$

$t=1$?

$$\left.\begin{array}{l} \lim\limits_{t \to 1^-} \varphi'(t) = \lim\limits_{t \to 1^-} (-1, 2-2t) = (-1,0) \\ \lim\limits_{t \to 1^+} \varphi'(t) = \lim\limits_{t \to 1^+} (-1,0) = (-1,0) \end{array}\right. \Rightarrow \varphi(t) \in C^1([0,2]).$$

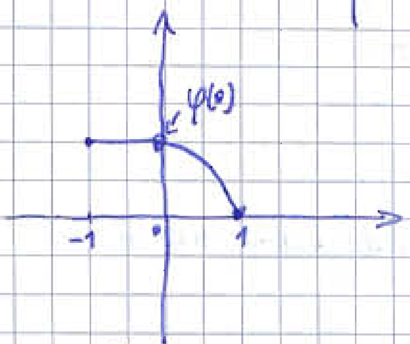
φ è regolare perchè $\varphi_1'(t) = -1 \neq 0 \Rightarrow \varphi'(t) \neq (0,0)$ $\forall t \in [0,2]$

Parametrizzare S con punto iniziale $(-1,1)$ e finale $(1,0)$.

$y=1 \quad x \in [-1,0[$

$\varphi(t) = (t, 1) \quad t \in [-1,0[\quad$ infatti $\varphi(-1) = (-1,1)$ PUNTO INIZIALE

$\varphi(0) = (0,1)$



$y = 1-x^2 \quad x \in [0,1]$

$\varphi(t) = (t, 1-t^2) \quad t \in [0,1] \qquad \varphi(0) = (0,1) = \varphi(0)$

$\varphi(1) = (1,0)$ PUNTO FINALE

$\gamma(t) = \begin{cases} (t,1) & t \in [-1,0[\\ (t, 1-t^2) & t \in [0,1] \end{cases}$ Metto assieme i due tratti.

Queste due parametrizzazioni sono equivalenti?

$\gamma : [-1,1] \to \mathbb{R}^2 \qquad \varphi : [0,2] \to \mathbb{R}^2 \qquad g(t) = (1-t) \qquad g : [-1,1] \to 2 \begin{array}{l} C^1 \\ g'(t) = -1 \neq 0 \\ \text{biettiva} \end{array}$

$$\gamma(g(t)) = \begin{cases} (1-t, 1) & g(t) \in [-1,0[\\ (1-t, 1-(1-t)^2) & g(t) \in [0,1] \end{cases} = \underbrace{\begin{cases} (1-t,1) & t \in]1,2] \\ (1-t, 2t-t^2) & t \in [0,1] \end{cases}}_{\varphi(t)}$$

$-1 \leq 1-t \leq 0$

$-2 \leq -t < -1$

$2 \geq t > 1$

$\Rightarrow$ Le due parametrizzazioni sono equivalenti

DEF.

Data $\varphi : I \to \mathbb{R}^2$ una curva regolare in $t_0 \in I$, allora il vettore

$$T(t_0) = \frac{\varphi'(t_0)}{|\varphi'(t_0)|}$$ è detto VERSORE TANGENTE a $\varphi(t)$ in $\varphi(t_0)$.

norma $\overset{\text{di}}{1}$. $\qquad T(t_0) = \dfrac{(\varphi_1'(t_0), \varphi_2'(t_0))}{|\varphi'(t_0)|}$

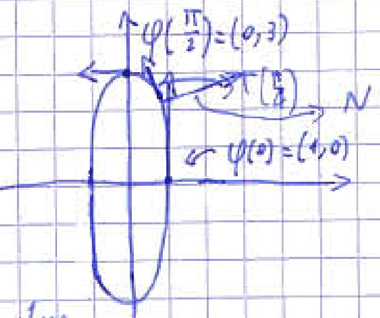
Il vettore $N(t_0) = \dfrac{\varphi_2'(t_0), -\varphi_1'(t_0)}{|\varphi'(t_0)|}$ è detto VERSORE NORMALE.

$\varphi(t) = (\cos t, 3 \operatorname{sen} t)$ $t \in [0, 2\pi]$ φ è una curva regolare, di classe C^1, chiusa e semplice.

ELLISSE

$$\begin{cases} x = \cos t \\ y = 3 \operatorname{sen} t \end{cases} \qquad \begin{cases} x = \cos t \\ \dfrac{y}{3} = \operatorname{sen} t \end{cases}$$

$x^2 + \left(\dfrac{y}{3}\right)^2 = 1$ ellisse



$\varphi'(t) = (-\operatorname{sen} t, 3\cos t)$

$\varphi'(0) = (0, 3)$ Il modulo dei due

$\varphi'(\frac{\pi}{2}) = (-1, 0)$ vettori è diverso perché, per la legge di Keplero, i corpi in movimento devono spazzare le stesse aree nello stesso tempo, quindi se il punto è più vicino al centro, si muoverà più velocemente.

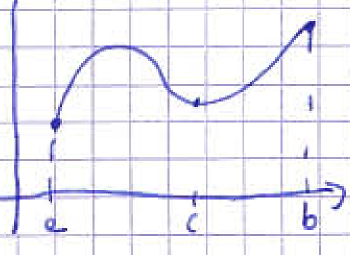
$\varphi'(\frac{\pi}{4}) = \left(-\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2}\right)$ $|\varphi'(\frac{\pi}{4})| = \sqrt{\frac{1}{2} + \frac{9}{2}} = \sqrt{5}$

$T(\frac{\pi}{4}) = \dfrac{\left(-\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2}\right)}{\sqrt{5}} = \left(-\dfrac{\sqrt{2}}{2\sqrt{5}}, \dfrac{3\sqrt{2}}{2\sqrt{5}}\right)$ $N(\frac{\pi}{4}) = \left(\dfrac{3\sqrt{2}}{2\sqrt{5}}, \dfrac{\sqrt{2}}{2\sqrt{5}}\right)$

T e N sono ortogonali. N risulta puntare verso l'esterno (scelto così).

$\varphi(t) = (1+t, 1+2t)$ $t \in [0,1]$ $\mathcal{L}_\varphi = d((1,1),(2,3)) = \sqrt{1+4} = \sqrt{5}$

Prendo una qualsiasi curva



La lunghezza della curva, in prima approssimazione, potrebbe essere uguale a $d(\varphi(a), \varphi(b))$. La lunghezza sarà sicuramente $>$.

Secondo passaggio: prendo il punto di mezzo e prendo la somma delle due distanze. $\mathcal{L}(\varphi) \geq d(\varphi(a), \varphi(c)) + d(\varphi(c), \varphi(b))$.

Via via, si potrebbero prendere tanti punti e approssimare sempre di più

$A = \{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$ Considero la poligonale $\{P_0 = \varphi(t_0), \ldots, P_n = \varphi(t_n)\}$

$\mathcal{L}(\varphi_A) = \sum_{i=1}^{n} d(P_i, P_{i-1})$ $\sup_A \mathcal{L}(\varphi_A) = \mathcal{L}(\varphi)$ per definizione

TEOREMA DI RETTIFICABILITÀ

Data $\varphi : [a,b] \to \mathbb{R}^2$ $[\mathbb{R}^3 \dots \mathbb{R}^n]$ di classe C^1, allora

① $\mathcal{L}(\varphi) < +\infty$

② $\mathcal{L}(\varphi) = \int_a^b |\varphi'(t)|\, dt$

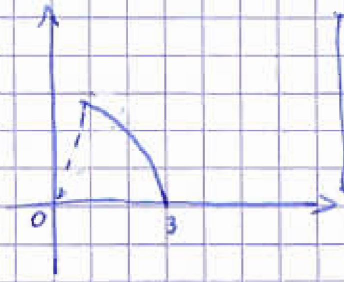
Le parametrizzazioni equivalenti lasciano invariata la lunghezza di una curva.

$\varphi(t) = (1+t, 1+2t)$ $t \in [0,1]$ $\varphi'(t) = (1,2)$ $|\varphi'(t)| = \sqrt{5}$

$\int_0^1 |\varphi'(t)|\, dt = \int_0^1 \sqrt{5}\, dt = \left[\sqrt{5}\, t\right]_0^1 = \sqrt{5}$ come trovato prima.

$\varphi(t) = (3\cos t, 3\,\mathrm{sen}\, t)$ $t \in \left[0, \frac{\pi}{3}\right]$

$\mathcal{L}(\varphi) = \frac{\pi}{3} \cdot 3 = \pi$



LUNGHEZZA ARCO
⇓
RAGGIO X ANGOLO

$\varphi'(t) = (-3\,\mathrm{sen}\, t, 3\cos t)$ $|\varphi'(t)| = 3$

$\mathcal{L}(\varphi) = \int_0^{\pi/3} 3\, dt = \frac{\pi}{3} \cdot 3 = \pi$

$\rho = \theta$ $\theta \in [0, 2\pi]$ ← spirale $\mathcal{L}(\varphi) = ?$

$\varphi = \begin{cases} x = \theta\cos\theta \\ y = \theta\,\mathrm{sen}\,\theta \end{cases}$ $\theta \in [0, 2\pi]$ $\varphi'(\theta) = (\cos\theta - \theta\,\mathrm{sen}\,\theta, \mathrm{sen}\,\theta + \theta\cos\theta)$

$|\varphi'(\theta)| = \sqrt{\underbrace{\cos^2\theta + \theta^2\mathrm{sen}^2\theta - \cancel{2\theta\,\mathrm{sen}\,\theta\cos\theta} + \mathrm{sen}^2\theta}_{1} + \theta^2\cos^2 t + \cancel{2\theta\,\mathrm{sen}\,\theta\cos\theta}} = \sqrt{\theta^2 + 1}$

$\mathcal{L}(\varphi) = \int_0^{2\pi} \sqrt{1+t^2}\, dt = t\sqrt{1+t^2} - \int \frac{t}{2\sqrt{1+t^2}} \cdot 2t\, dt = t\sqrt{1+t^2} - \int t^2 \cdot \frac{1}{\sqrt{1+t^2}}\, dt \dots$

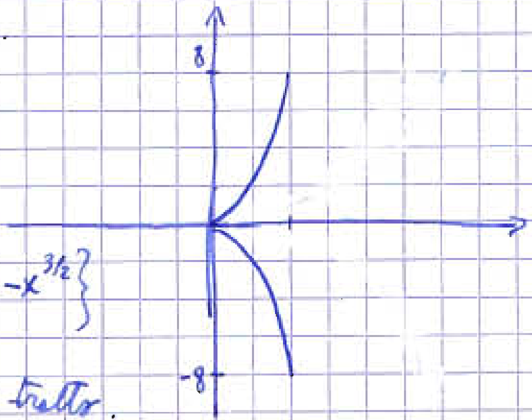
$\varphi(t) = (t^2, t^3) \quad t \in [-2,2]$ φ curva di classe C^1

$\varphi(-2) \neq \varphi(2)$ non è chiusa $\quad \varphi'(t) = (2t, 3t^2)$ curva regolare a tratti $\varphi'(0) = (0,0)$

e $\varphi(t) \neq (0,0) \quad \forall t \neq (0,0)$

φ è semplice perché $\varphi_2 = t^3$ è iniettiva.

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases} \quad \begin{cases} t = \sqrt[3]{y} \\ x(t) = y^{\frac{2}{3}} \end{cases} \quad y \in [-8,8]$$



$$S = \{(x,y) : x \in [0,4],\ y = x^{3/2}\} \cup \{(x,y) : x \in [0,4],\ y = -x^{3/2}\}$$

$\mathcal{L}(\varphi)$ sarà 2 · lunghezza del primo tratto.

$$|\varphi'(t)| = \sqrt{4t^2 + 9t^4} = |t| \sqrt{4 + 9t^2} = 2|t| \sqrt{1 + \frac{9}{4}t^2}$$

$$\mathcal{L}(\varphi) = \int_{-2}^{2} 2|t| \sqrt{1 + \frac{9}{4}t^2}\, dt = \underset{\nearrow}{2} \int_0^2 2t \sqrt{1 + \frac{9}{4}t^2}\, dt = 2\frac{8}{9}\int_0^2 \frac{9}{4} \cdot 2t \sqrt{1 + \frac{9}{4}t^2}\, dt =$$

due volte il tratto da 0 a 2 per simmetria.

$$= \frac{16}{9} \left[\frac{\left(1 + \frac{9}{4}t^2\right)^{3/2}}{3/2} \right]_0^2 = \frac{16}{9} \cdot \left[\frac{2}{3} \cdot \left(10^{3/2} - 1\right) \right] = 36,29$$

CALCOLO LUNGHEZZA CURVA DA COORDINATE POLARI

$$\rho(\theta) = f(\theta) \quad \theta \in [a,b]$$
$$\Downarrow$$

$$\begin{cases} x(\theta) = f(\theta) \cos\theta \\ y(\theta) = f(\theta) \operatorname{sen}\theta \end{cases} \quad \theta \in [a,b] \quad \begin{cases} x'(\theta) = f'(\theta) \cos\theta - f(\theta) \operatorname{sen}\theta \\ y'(\theta) = f'(\theta) \operatorname{sen}\theta + f(\theta) \cos\theta \end{cases}$$

$$|\varphi'(\theta)| = \sqrt{x(\theta)^2 + y(\theta)^2} = \dots = \sqrt{(f'(\theta))^2 + (f(\theta))^2}$$

$$\mathcal{L}(\varphi) = \int_a^b \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, dt \quad \text{se} \quad \rho = f(\theta)$$

Se invece $y = f(x)$, cioè $\varphi(t) = (t, f(t)) \quad t \in [\alpha, \beta]$

$$|\varphi'(t)| = \sqrt{1 + f'(t)^2} \qquad \mathcal{L}(\varphi) = \int_\alpha^\beta \sqrt{1 + f'(t)^2}\, dt$$

INTEGRALE CURVILINEO

$\varphi : [a,b] \to \mathbb{R}^2 \quad C^1$

$f : \varphi([a,b]) \to \mathbb{R}$

$$\int_\varphi f\, ds \underset{\underline{\underline{DEF}}}{\overset{\uparrow}{=}} \int_a^b f(\varphi(t)) \cdot \underbrace{|\varphi'(t)|\, dt}_{ds}$$

Un'applicazione è il calcolo del baricentro della curva, cioè il punto di sospensione.

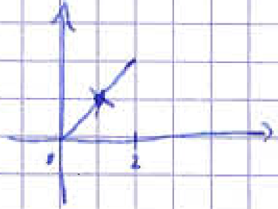
$$X = \frac{\int x \cdot f\, ds}{\int f\, ds} \quad \begin{matrix} \text{MOMENTO RISPETTO ASSE X} \\ \text{MASSA} \end{matrix}$$

DISTANZA

$$Y = \frac{\int y \cdot f\, ds}{\int f\, ds}$$

Se $f = 1$, si parla di BARICENTRO GEOMETRICO

ESEMPIO

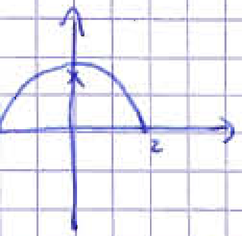


$\mathcal{L}(\varphi_1) = 2\sqrt{2}$

$f = 1$

$$\int_{\varphi_1} x\, ds = \int_0^2 \sqrt{1+1} \cdot t\, dt = \left[\frac{\sqrt{2}}{2} t^2\right]_0^2 = 2\sqrt{2} - 0 \qquad x = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

$$\int_{\varphi_1} y\, ds = \int_0^2 \sqrt{2} \cdot t\, dt = \left[\frac{\sqrt{2}}{2} t^2\right]_0^2 = 2\sqrt{2} - 0 \qquad y = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$



$\mathcal{L}(\varphi_2) = 2\pi$

$f = 1 \qquad \varphi_2' = (-2\text{sen}t, 2\cos t)$

$$\int_{\varphi_2} x\, ds = \int_0^\pi \sqrt{(-2\text{sen}t)^2 + (2\cos t)^2} \cdot 2\cos t = $$

$$= \int_0^\pi 4\cos t = 4\left[\text{sen}t\right]_0^\pi = 0$$

$$\int_\varphi y\, ds = \int_0^\pi |\varphi_2'(t)| \cdot 2\text{sen}t\, dt = \int_0^\pi 4\text{sen}t = 4\left[-\cos t\right]_0^\pi = 4 - 4(-1) = 8$$

$$y = \frac{8}{\underset{\nwarrow \text{lunghezza}}{2\pi}} = \frac{4}{\pi} \qquad x = 0$$

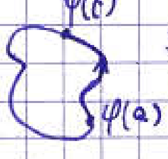
17/03/08

ASCISSA CURVILINEA

$t \in [a,b] \quad \varphi \in C^1$

Dà un riferimento sopra la curva.

$$s(t) = \int_a^t |\varphi'(\sigma)|\, d\sigma$$



$$s(t) = \int_a^t |\varphi'(\sigma)|\, d\sigma$$

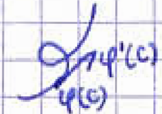
$$\frac{ds(t)}{dt} = |\varphi'(t)| \qquad ds = |\varphi'(t)|\, dt$$

CURVE IN $\mathbb{R}^3$

$\varphi: [a,b]=I \to \mathbb{R}^3$ φ è una curva se I è un intervallo e φ è continua

$$\varphi: \begin{cases} x(t)=\varphi_1(t) \\ y(t)=\varphi_2(t) \\ z(t)=\varphi_3(t) \end{cases} \qquad \varphi'(t)=\left(\varphi_1'(t), \varphi_2'(t), \varphi_3'(t)\right)$$

- CURVA CHIUSA $\varphi(a)=\varphi(b)$
- CURVA SEMPLICE $\forall t_1, t_2$ t.c. $t_1 \neq t_2 \Rightarrow \varphi(t_1) \neq \varphi(t_2)$
- CURVE EQUIVALENTI trovare $g: I \to J$ t.c. $\varphi(g(t)) \Rightarrow \varphi$
- CURVA REGOLARE $\varphi'(t) \neq (0,0,0)$
- VERSORE TANGENTE $= \mathbb{R}^2$



- VERSORE NORMALE non lo definiamo (infiniti vettori ∞^2)
- BARICENTRO GEOMETRICO

$$x_B = \frac{\int_\varphi x\,ds}{\mathcal{L}(\varphi)} \qquad y_B = \frac{\int_\varphi y\,ds}{\mathcal{L}(\varphi)} \qquad z_B = \frac{\int_\varphi z\,ds}{\mathcal{L}(\varphi)}$$

- BARICENTRO CON DENSITÀ DI MASSA

$$x_B = \frac{\int_\varphi x\, f\, ds}{\int_\varphi f\, ds} \qquad y_B = \frac{\int_\varphi y \cdot f\, ds}{\int_\varphi f \cdot ds} \qquad z_B = \frac{\int_\varphi z \cdot f\, ds}{\int_\varphi f \cdot ds}$$

ELICA CILINDRICA

$\varphi: [0,2\pi] \to \mathbb{R}^3$ $\varphi(t)=(\cos t, \operatorname{sen} t, t)$ φ è una curva.

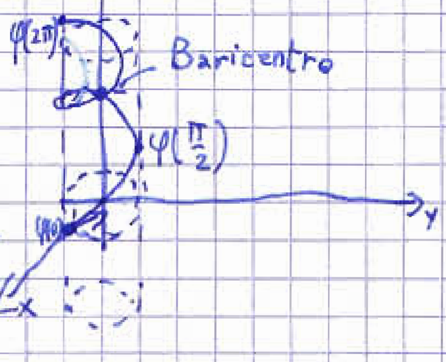
$\varphi'(t)=(-\operatorname{sen} t, \cos t, 1)$ continua $\Rightarrow \varphi \in C^1$ e $\varphi_3'(t)=1 \neq 0 \Rightarrow \varphi$ è regolare

$\varphi_3(t)=t$ è iniettiva $\Rightarrow \varphi$ è semplice.

$\varphi(0) \neq \varphi(2\pi) \Rightarrow \varphi$ non è chiusa.

$$\begin{cases} x=\cos t \\ y=\operatorname{sen} t \\ z=t \end{cases} \Rightarrow \begin{cases} x^2+y^2=1 \\ z=t \end{cases}$$

La curva sta sul cilindro con base una cfr di raggio 1 centrata in O



$\varphi(0)=(1,0,0)$

$\varphi\left(\frac{\pi}{2}\right) = \left(0, 1, \frac{\pi}{2}\right) \quad \varphi(\pi) = (-1, 0, \pi) \quad \varphi\left(\frac{3}{2}\pi\right) = \left(0, -1, \frac{3}{2}\pi\right) \quad \varphi(2\pi) = (1, 0, 2\pi)$

$\varphi'(0) = (0, 1, 1) \quad \mathcal{L}(\varphi) = \int_0^{2\pi} |\varphi'(t)|\, dt = \int_0^{2\pi} \sqrt{(-\text{sen}t)^2 + (\cos t)^2 + 1^2} = \int_0^{2\pi} \sqrt{\text{sen}^2 t + \cos^2 t + 1}\, dt =$

$= \int_0^{2\pi} \sqrt{2}\, dt = \left[\sqrt{2}t\right]_0^{2\pi} = \sqrt{2} \cdot 2\pi$

L'asse z è di simmetria ⇒ il baricentro starà su z ⇒ $x_B = 0$, $y_B = 0$

$\int_\varphi x\, ds = \int_0^{2\pi} \cos t \cdot |\varphi'(t)|\, dt = \int_0^{2\pi} \cos t \cdot \sqrt{2}\, dt = \left[\sqrt{2}\,\text{sen}t\right]_0^{2\pi} = 0 - 0 = 0$

x è la funzione $\mathbb{R}^3 \to \mathbb{R}$
$(x,y,z) \to x$

$\int_\varphi y\, ds = \int_0^{2\pi} \text{sen}t \cdot \sqrt{2}\, dt = \left[-\sqrt{2}\cos t\right]_0^{2\pi} = -\sqrt{2} - (-\sqrt{2}) = 0$

$\int_\varphi z\, ds = \int_0^{2\pi} t \cdot \sqrt{2}\, dt = \left[\sqrt{2}\frac{t^2}{2}\right]_0^{2\pi} = \frac{\sqrt{2}}{2} \cdot (2\pi)^2 - 0 = \frac{\cancel{4}^{2}\sqrt{2}\,\pi^2}{\cancel{2}} = 2\sqrt{2}\,\pi^2$

$x_B = 0 \quad y_B = 0 \quad z_B = \frac{\cancel{2\sqrt{2}}\,\pi^{\cancel{2}}}{} = \pi$

lunghezza → $\sqrt{2} \cdot \cancel{2}\pi$

BARICENTRO DI MASSA

$f(x,y,z) = z$ a $z=0$, i punti "pesano" poco; a $z = 2\pi$ "pesano" molto.

Ci aspettiamo che x_B e y_B non cambino

$\int_\varphi f\, ds = \int_0^{2\pi} \underbrace{f(\varphi(t))}_{\varphi_3(t)} \cdot |\varphi'(t)|\, dt = \int_0^{2\pi} t \cdot \sqrt{2}\, dt = \left[\sqrt{2}\frac{t^2}{2}\right]_0^{2\pi} = 2\sqrt{2}\,\pi^2$

$\int_\varphi \underset{\uparrow}{x}\, \underset{\updownarrow}{f}\, \underset{\Uparrow}{ds} = \int_0^{2\pi} \underset{\uparrow}{\cos t} \cdot \underset{\updownarrow}{f(\varphi(t))} \cdot \underset{\Uparrow}{|\varphi'(t)|}\, dt = \int_0^{2\pi} \underset{\uparrow}{\cos t} \cdot \underset{\updownarrow}{t} \cdot \underset{\Uparrow}{\sqrt{2}}\, dt = \sqrt{2} \int_0^{2\pi} t\cos t\, dt$

$\int t\cos t\, dt = t \cdot (\text{sen}t) - \int (-\text{sen}t) \cdot 1\, dt = t\,\text{sen}t + (-\cos t) = t\,\text{sen}t - \cos t$

$\sqrt{2}\int_0^{2\pi} t\cos t\, dt = \sqrt{2}\left[t\,\text{sen}t - \cos t\right]_0^{2\pi} = \sqrt{2}(2\pi \cdot 0 - 1) - \sqrt{2}(0 - 1) = -\sqrt{2} + \sqrt{2} = 0$

$\int_\varphi y f\, ds = \int_0^{2\pi} \text{sen}t \cdot t \cdot \sqrt{2}\, dt = \sqrt{2}\left(\int_0^{2\pi} t\,\text{sen}t\, dt\right) = \sqrt{2}\left(t(-\cos t) - \int_0^{2\pi} 1 \cdot (-\cos t)\, dt\right) =$

$= \sqrt{2}\left[-t\cos t + \text{sen}t\right]_0^{2\pi} = \sqrt{2}\left(-2\pi + 0 - (0+0)\right) = -2\sqrt{2}\,\pi$

$\int z f\, ds = \int_0^{2\pi} t \cdot t \cdot \sqrt{2}\, dt = \sqrt{2}\left[\frac{t^3}{3}\right]_0^{2\pi} = \sqrt{2}\left(\frac{8\pi^3}{3} - 0\right) = \frac{8\sqrt{2}\,\pi^3}{3}$

$x_B = 0$

$$y_B = \frac{-2\sqrt{2}\pi}{2\sqrt{2}\pi^2} = -\frac{1}{\pi} \qquad z_B = \frac{8\sqrt[4]{2}\pi^3}{3} \cdot \frac{1}{2\sqrt{2}\pi^2} = \frac{4}{3}\pi$$

FUNZIONI DI PIÙ VARIABILI

$f : \Omega \subseteq \mathbb{R}^2 (\mathbb{R}^3) \to \mathbb{R}$ è una funzione quando è assegnato $\Omega \subseteq \mathbb{R}^2 (\mathbb{R}^3) \neq \emptyset$, è assegnata la legge f.

OSS.

Se è assegnata la sola legge "f" ha senso considerare il dominio di f il più grande insieme su cui può agire $f : \{(x,y) \in \mathbb{R}^2 : f(x,y) \in \mathbb{R}\}$.

ESEMPIO

$f(x,y) = \log(1-x^2-y^2)$ è necessario che $1-x^2-y^2 > 0$, cioè la palla di centro $0,0$ e raggio 1. $(x,y) \in B(0,0;1) = \{(x,y) : x^2+y^2 < 1\}$.

$\text{dom}(f) = B(0,0;1)$

$f(x,y) = \sqrt{x^2+y^2-2x-2}$ è necessario che $x^2+y^2-2x-2 \geq 0$

$(x^2-2x+1)-1+y^2-2 = (x-1)^2+y^2-3 \geq 0 \quad \text{dom}(f) = \{(x,y) : (x-1)^2+y^2 \geq 3\}$

Complementare di una sfera di centro $(1,0)$ e raggio $\sqrt{3}$.

$$f(x,y) = x \qquad f : \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \to x \end{array} \qquad g(x,y) = y \qquad g : \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \to y \end{array}$$

Funzioni continue (f e g) su $\mathbb{R}^2$

TEOREMA

Composizione, rapporto, prodotto, quoziente, somma di funzioni continue è continua (stessa dimostrazione del caso di 1 variabile).

$f(x,y) = \sqrt{1+\log(3+\text{sen}\, xy)}$ dom $= ?$

$4 \geq 3+\text{sen}\, xy \geq 2 \quad \forall x,y \quad \log(3+\text{sen}\, xy)$ è monotona $\Rightarrow \log(4) \geq \log(3+\text{sen}\, xy) \geq \log($

$\Rightarrow 1+\log(4) \geq 1+\log(3+\text{sen}\, xy) \geq \log(2)+1 > 0$

$\Rightarrow \text{dom}(f) = \mathbb{R}^2$

$f(x,y)$ è continua su tutto il dominio perché $x\cdot y$, $\text{sen}(x\cdot y)$, $3+\text{sen}(x\cdot y)$, $\log(3+\text{sen}\,xy)$, $1+\log(3+\text{sen}\,xy)$ sono continue, quindi anche $\sqrt{1+\log(3+\text{sen}\,xy)}$.

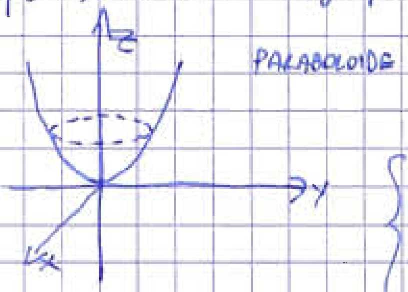
PUNTI DI MAX E MIN

Data $f: A \to \mathbb{R}^2$ $(x_0,y_0)=P_0\in A$ si dice

- PUNTO DI MINIMO relativo se $\exists\ \overset{\text{palla}}{B}(P_0,r)\subset A$ tale che $f(x,y)\geq f(P_0)$ $\forall (x,y)\in B(P_0,$
- PUNTO DI MASSIMO relativo se $\exists\ (P_0,r)\subset A$ tale che $f(x,y)\leq f(P_0)$ $\forall (x,y)\in B(P_0,$
- PUNTO DI MINIMO assoluto se $f(x,y)\geq f(P_0)$ $\forall (x,y)\in A$
- PUNTO DI MASSIMO assoluto se $f(x,y)\leq f(P_0)$ $\forall (x,y)\in A$.

ESEMPIO

$f(x,y)=x^2+y^2$ Grafico $(f)=\{(x,y,z): z=f(x,y)=x^2+y^2, (x,y)\in\mathbb{R}^2\}$



Provo a intersecare il grafico con i piani coordinati.

$$\begin{cases} z=x^2+y^2 \\ x=0 \end{cases} \quad \begin{cases} z=y^2 \\ x=0 \end{cases} \qquad \begin{cases} z=x^2+y^2 \\ y=0 \end{cases} \quad \begin{cases} z=x^2 \\ y=0 \end{cases}$$

Provo a tagliare $f(x,y)$ con $z=K$ (insiemi di livello)

$\{f=K\}=\{(x,y): f(x,y)=K\}=\{(x,y): x^2+y^2=K\}$ INSIEMI DI LIVELLO

$K<0$ insieme vuoto $\{f=K\}=\emptyset$

$K=0$ $\{f=0\}=(0,0)\Rightarrow$ punto di minimo assoluto perché sotto non c'è nulla.

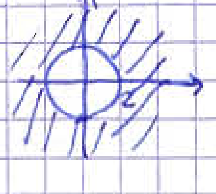
$K>0$ $x^2+y^2=(\sqrt{K})^2$ circonferenza di raggio $\sqrt{K}$

$\{f=K\}=\{(x,y), x^2+y^2=K\}=\underbrace{\partial B(0,0;\sqrt{K})}_{\text{bordo della palla}}$

$\{f\geq K\}=$ SOPRALIVELLI $=\{(x,y): f(x,y)\geq K\}$

$\{f\leq K\}=$ SOTTOLIVELLI $=\{(x,y): f(x,y)\leq K\}$

$\{f\geq 2\}=\{(x,y): x^2+y^2\geq 2\}$

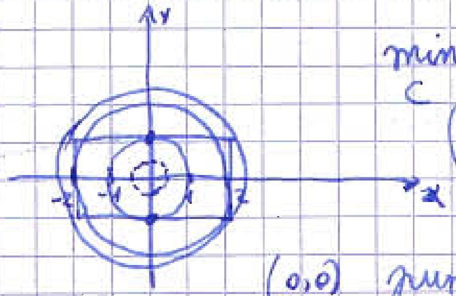


$\{f\geq -3\}$

Determinare massimi e minimi relativi e assoluti di $f=x^2+y^2$ su

$\underset{x}{[-2,2]} \times \underset{y}{[-1,1]} = C$

$\exists$ max e min assoluti poichè f continua su C "chiuso" e "limitato" (Weierstrass)



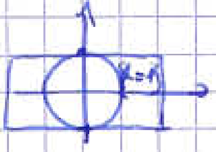
$\min_C f = f(0,0) = \min_{\mathbb{R}^2} f$ (minimo assoluto)

$\left(\min_{\mathbb{R}^2} f \leq \min_C f\right.$ quando $\left. C \subseteq \mathbb{R}^2\right)$

(0,0) punto di minimo assoluto di f su C. È anche punto di minimo relativo perchè $\exists B(0,0;\frac{1}{2}) : \forall (x,y) \in B(0,0;\frac{1}{2})\ f(x,y) \geq f(0,0)$

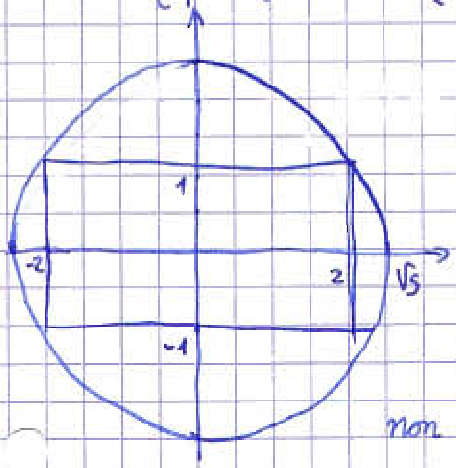
Per il massimo, disegno vari insiemi di livello più larghi fino a toccare gli estremi.

$k=1 \to \{f=k\} \cap \partial C = \{(0,1),(0,-1)\}$



metodo visivo

$k=5 \to \{f=5\} \cap \partial C = \{(2,1),(2,-1),(-2,1),(-2,-1)\}$



$\forall k > 5\ \{f=k\} \cap C = \emptyset$

$\Rightarrow$ 5 è il massimo di f su C

f(1,2) punto di massimo assoluto, ma non relativo perchè se prendo una palla centrata in (1,2) non riuscirà mai a stare in f.

ESERCIZIO

Cerca max e min di $y=x^2+y^2$ su

$\to [1,3] \times [-2,2]$

$\to \{(x+1)^2+y^2<1\}$

$\to [2,3] \times [2,3]$

TOPOLOGIA

DEF $P_0 \in \mathbb{R}^2$, dato $A \subseteq \mathbb{R}^2$

A si dice INTORNO DI P_0 se $\exists\, B(P_0,R) \subset A$

$B(P_0,R) = \{(x,y) : d(x,y;P_0) < R\}$ intorno di P_0

P_1 è esterno a Ω

$\exists\, B(P_1,R) \subset \Omega^c$

$\exists\, B(P_0,R) \subset \Omega$ $\quad P_0$ è interno a

$\bar{P} \in \partial\Omega \equiv$ frontiera di Ω, cioè

$\forall R \quad B(\bar{P},R) \cap \Omega \neq \emptyset$ e $B(\bar{P},R) \cap \Omega^c \neq 0$.

palla centro raggio

DEF Un insieme $C \subset \mathbb{R}^2$ è chiuso se $\partial C \subset C$

Un insieme $A \subseteq \mathbb{R}^2$ è aperto se A^c è chiuso

$P \in \mathbb{R}^2$ si dice PUNTO DI ACCUMULAZIONE per A se $\forall R > 0 \quad B(P,R) \cap (A \setminus P) \neq \emptyset$

$P \in A$, se P non è di accumulazione, P è ISOLATO

ESEMPIO

$\Omega = [0,1] \times [0,1]$ è chiuso, infatti la frontiera è fatta da 4 segmenti che stanno tutti in Ω: $\delta\Omega = \{1\}\times[0,1] \cup \{0\}\times[0,1] \cup [0,1]\times\{1\} \cup [0,1]\times\{0\}$

$[0,1[\,\times\,[0,1[$ non è aperto, non è chiuso.

Al complementare mancano gli altri due tratti

$(1,1) \in \delta\Omega$ e $(1,1) \notin \Omega$

$\Rightarrow \Omega$ non è chiuso

$(0,0) \in \delta\Omega^c$ e $(0,0) \notin \Omega^c \Rightarrow \Omega^c$ non è chiuso $\Rightarrow \Omega$ non è aperto.

$(1,1)$ è di accumulazione per Ω $((1,1) \notin \Omega)$

perchè la pallina contiene elementi dell'insieme.

$(0,0)$ è di accumulazione per Ω e $(0,0) \in \Omega$.

OSS Se P è interno a Ω, allora P è di accumulazione.

OSS A aperto $\Leftrightarrow$ A è intorno di ogni suo punto $\Rightarrow$ ogni punto di A è di accumulazione per A.

OSS $\delta\Omega^c = \delta\Omega$

DEF. (limite)

Data $f: A \to \mathbb{R}$ $A \subseteq \mathbb{R}^2$, P_0 punto di accumulazione per A

$\exists \lim_{P \to P_0} f(P) = l \in \mathbb{R} \cup \{+\infty, -\infty\}$ SE

$\forall J \in \mathcal{U}_l \ \exists H \in \mathcal{U}_{P_0} : f(x,y) \in J \ \forall (x,y) \in H \cap (A \setminus P_0)$

Nel caso di funzioni di una variabile era:

$\lim_{x \to 0} \frac{\text{sen} x}{x} = 1$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \neq 0 \ -\delta < x < \delta \ |f(x) - 1| < \varepsilon$ cioè

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in (\underbrace{\mathbb{R} \setminus \{0\}}_{\text{Dom } f(x)}) \cap (]-\delta, \delta[\setminus \{0\}) \ 1 - \varepsilon < f(x) < 1 + \varepsilon$ cioè

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in (\mathbb{R} \setminus \{0\}) \cap (]-\delta, \delta[\setminus \{0\}) \ f(x) \in]1 - \varepsilon, 1 + \varepsilon[$

$\forall J_\varepsilon \ \exists H_\delta : f(x) \in J_\varepsilon \ \forall x \in \text{dom} f \cap (H_\delta - \{0\})$ ← due variabili

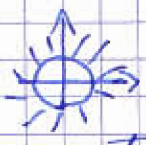
Nel caso di due variabili, non ha senso $\lim_{x,y \to +\infty} \ldots$, mentre ha senso $\lim_{\|(x,y)\| \to +\infty} \ldots$, cioè andiamo verso $+\infty$ in ogni direzione.

DEF.

$A \subseteq \mathbb{R}^2$ è detto INTORNO DI INFINITO se $\exists R > 0 : B(0,0;R) \subset A$

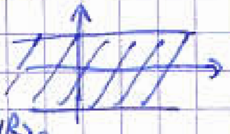
esempio

$A_1 = \{(x,y) : x^2 + y^2 \geq 4\}$ è intorno di ∞



Contiene tutte le direzioni che portano a ∞

$A_2 \{(x,y) : -1 \leq y \leq 1\}$ non è intorno di ∞



Infatti A_2 non contiene $\{(x,y) : x^2 + y^2 \geq R^2\}$ $\forall R > 0$

In una variabile $]a, +\infty[\equiv$ intorno di $+\infty$, $]-\infty, b[\equiv$ intorno di $-\infty$

$|x| > c > 0 \equiv$ intorno di ∞

$]-\infty, -c] \cup [c, +\infty[$

esempio

$\lim_{\|x,y\| \to \infty} \frac{x}{x+y} \nexists$ verificare...

TEOREMA: se il limite esiste, allo
è unico

Abbiamo bisogno di alcune nozioni!

DEF Data $f: A \to \mathbb{R}$, $B \subseteq A$ diciamo RESTRIZIONE DI f A B la funzione

$f_{|B} : B \to \mathbb{R}$ $\quad f_{|B}(x,y) = f(x,y) \quad \forall (x,y) \in B$

esempio di restrizione

$f(x,y) = x+3y$ dom $f = \mathbb{R}^2$ $\quad \varphi(t) = (t,t) \quad t \in \mathbb{R}$

$f_{|\varphi(t)}(x,y) = f(\varphi(t)) = t+3t = 4t$ Leggo la funzione solo nei punti della retta

$\lim_{t\to+\infty} f_{|\varphi} = +\infty$ Le restrizioni mi permettono di passare a limiti di 1 variabile.

$f(x,y) = x\log(x^2+y^2) \quad \lim_{(x,y)\to(0,0)} f(x,y)$?

$\varphi(t) = (t,t)$

$f_{|\varphi(t)} = f(\varphi(t)) = t\log 2t^2 \xrightarrow[t\to0]{} 0^-$ $\quad \gamma(t) = (0,t)$

$f_{|\gamma(t)} = f(\gamma(t)) = 0\cdot\log(0^2+t^2) = 0 \xrightarrow[t\to0]{} 0$ $\quad \varphi(t) = (t,0)$

$f_{|\varphi(t)} = f(\varphi(t)) = t\cdot\log t^2 \xrightarrow[t\to0]{} 0^-$ $\quad \gamma_1(t) = (t,t^2)$

$f_{|\gamma_1(t)} = f(\gamma_1(t)) = t\log(t^2+t^4) \xrightarrow[t\to0]{} 0^-$

$\Longrightarrow \lim_{(x,y)\to(0,0)} x\log(x^2+y^2) = 0.$

$\forall\varepsilon>0 \ \exists B(0,0;R) : |x\log(x^2+y^2)| < \varepsilon \quad \forall x,y \in B(0,0;R)\setminus\{(0,0)\}$

$\forall\varepsilon>0 \ \exists R>0 : |x\log(x^2+y^2)| < \varepsilon \quad \forall x,y : 0 < x^2+y^2 < R^2$

$|x\log(x^2+y^2)| \le \sqrt{x^2+y^2}\log(x^2+y^2)$

$\forall\varepsilon>0 \ \exists R=\varepsilon : |x\log(x^2+y^2)| < |R\log R^2| = \varepsilon\log\varepsilon^2 \xrightarrow[\varepsilon\to0]{} 0$

TEOREMA

Se $\exists$ il limite di $f(x,y)$ per $(x,y)\to P_0$ e vale l allora $\exists \lim_{t\to t_0} f(\varphi(t)) = l$ $\forall \varphi : [t_0, t_0+\delta[\to \mathbb{R}^2 : \varphi(t_0) = P_0$.

OSS

Date $\varphi(t)$ e $\gamma(t)$ due curve con $\varphi(t_0) = P_0$ e $\gamma(t_0) = P_0$ tale che $\exists \lim_{t\to t_0} f(\gamma(t)) = l_1$ e $\exists \lim_{t\to t_0} f(\varphi(t)) = l_2$, con $l_1 \ne l_2 \Rightarrow \nexists \lim_{(x,y)\to P_0} f(x,y)$

ESERCIZIO

$\nexists \lim_{\|(x,y)\| \to \infty} \frac{x}{x+y}$ infatti presa $\varphi(t) = (t,0)$ $\lim_{t \to \infty} f(\varphi(t)) = \lim_{t \to \infty} \frac{t}{t+0} = 1$

$\gamma(t) = (0,t)$ $\lim_{t \to \infty} f(\gamma(t)) = \lim_{t \to \infty} \frac{0}{0+t} = 0 \neq 1 \Rightarrow \nexists \lim_{\|(x,y)\| \to \infty} \frac{x}{x+y}$

$\lim_{\|(x,y)\| \to \infty} x^2+y^4+3x-y-5$ $\gamma(t) = (t,0)$ $f(\gamma(t)) = t^2+3t-5 \xrightarrow[t \to \infty]{} +\infty$

Vogliamo stimare $f(x,y) \geq g(\rho)$: $\lim_{\rho \to +\infty} g = +\infty$ mettere g sotto f

$x^2+y^4+3x-y-5 \underset{x^2+y^2 \geq 1}{\geq} x^2+y^2+3x-y-5 \geq x^2+y^2-3|x|-|y|-5 \geq x^2+y^2-3\sqrt{x^2+y^2}-\sqrt{x^2+y^2}-5 =$

$= \rho^2-4\rho-5$ $\lim_{\rho \to +\infty} \rho^2-4\rho-5 = +\infty$

$f(x,y) \geq g(\rho) = \rho^2-4\rho-5$ $\forall \rho \geq 1$ $\forall \theta \in [0,2\pi]$

$\Rightarrow f(x,y)$ viene "sparato" a $+\infty \Rightarrow \lim_{\|(x,y)\| \to \infty} f(x,y) = +\infty$

TEOREMA

Data $f: A \to \mathbb{R}$, ∞ punto di accumulazione per A sono equivalenti:

(1) $\lim_{\|(x,y)\| \to \infty} f(x,y) = +\infty$

(2) $\exists g:]\rho_0, +\infty[$ tale che $f(x,y) \geq g(\rho)$ $\forall \rho \geq \rho_0$ $\forall \theta \in [0,2\pi]$ $\lim_{\rho \to \infty} g(\rho) = +\infty$

TEOREMA

Data $f: A \to \mathbb{R}$, $(0,0)$ di accumulazione per A sono equivalenti:

(1) $\lim_{(x,y) \to (0,0)} f = l \in \mathbb{R}$

(2) $\exists g:]0, \rho_0[$: $\lim_{\rho \to 0} g(\rho) = 0$ $|f(x,y) - l| \leq g(\rho)$

TEOREMA DI WEIERSTRASS

Data $f: C \to \mathbb{R}$ continua, con C chiuso e limitato, allora
$\exists P_m, P_M \in C$ tale che $f(P_m) =$ minimo $f(x,y)$ $(x,y) \in C$
$f(P_M) =$ massimo $f(x,y)$ $(x,y) \in C$

DEF

$C \subseteq \mathbb{R}^2$ si dice LIMITATO se $\exists P_0 \in \mathbb{R}^2$ $\exists R > 0$ tale che $C \subset B(P_0;R)$

COROLLARIO

Data $f: \mathbb{R}^2 \to \mathbb{R}$ continua tale che $\exists \lim_{\|(x,y)\| \to \infty} f(x,y) = l$

(1) $l = +\infty \Rightarrow \exists \min_{\mathbb{R}^2} f$

(2) $l = -\infty \Rightarrow \exists \max_{\mathbb{R}^2} f$

(3) $l = 0$ e $\exists \bar{P} : f(\bar{P}) > 0 \Rightarrow \exists \max_{\mathbb{R}^2} f$

(4) $l = 0$ e $\exists \bar{P} : f(\bar{P}) < 0 \Rightarrow \exists \min_{\mathbb{R}^2} f$

ESEMPI

$$f(x,y) = x^2+y^2+\log(1+x^2)+\underbrace{\operatorname{sen}(x^2+y^2)}_{\geq -1}$$

$$f(x,y) \geq \rho^2 - \log(1+\underset{+y^2}{x^2}) - 1 \geq \rho^2 - \log(1+\rho^2) - 1 \qquad \lim_{\rho \to \infty} \rho^2 - \log(1+\rho^2) - 1 = +\infty$$

$\Rightarrow \lim_{\|(x,y)\| \to \infty} f(x,y) = +\infty$ per il corollario del T. di Weierstrass, $\exists \min_{\mathbb{R}^2} f$

$$f(x,y) = \frac{x+\operatorname{sen}(x^2+y^2)}{x^2+y^2} \qquad |f(x,y)| \leq \frac{\sqrt{x^2+y^2}+1}{x^2+y^2} = \frac{\rho+1}{\rho^2} \xrightarrow[\rho \to +\infty]{} 0$$

valgono (3) e (4) perchè c'è sia un punto in cui la funzione è negativa sia uno in cui è negativa $\Rightarrow \exists$ min e max.

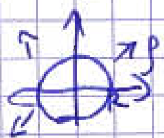
07/04/08

$\lim_{\|(x,y)\| \to \infty} f(x,y) = l \in \mathbb{R}$ è come dire

$\forall \varepsilon > 0 \quad \exists R > 0 : \forall (x,y) \in \mathbb{R}^2 \quad \underbrace{x^2+y^2}_{\rho^2} \geq R^2 \quad f(x,y) \in]l-\varepsilon, l+\varepsilon[$

$\forall \varepsilon > 0 \quad \exists R > 0 : \forall (\rho, \theta) \quad \rho \geq R \quad \forall \theta \in [0, 2\pi[\quad f(\rho\cos\theta, \rho\operatorname{sen}\theta) \in]l-\varepsilon, l+\varepsilon[$

$$\lim_{\substack{\rho \to \infty \\ \forall \theta}} f(\rho\cos\theta, \rho\operatorname{sen}\theta) = l$$



ESEMPIO

$-1 \leq \operatorname{sen}\theta \leq 1$

$-3\rho \leq 3\rho\operatorname{sen}\theta \leq 3\rho$ ← valore minimo

$3\rho \geq -3\rho\operatorname{sen}\theta \geq -3\rho$

$$\lim_{\|(x,y)\| \to \infty} (x^2+y^2+y^4+3x-3y) = \lim_{\substack{\rho \to \infty \\ \forall \theta}} (\rho^2 + \underset{\text{positivo}}{\rho^4\operatorname{sen}^4\theta} + 3\rho\underset{\geq -1}{\cos\theta} - 3\rho\underset{\geq -1}{\operatorname{sen}\theta}) \geq \lim_{\rho \to \infty} (\rho^2 - 3\rho - 3\rho) = +\infty$$

CALCOLO DIFFERENZIALE

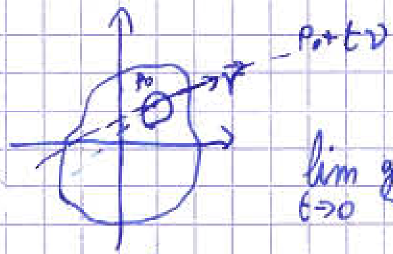
essendo A aperto $\exists B(P_0, R) \subset A$

DEF. data $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ aperto e dato $P_0 = (x_0, y_0) \in A$

$v = (v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 = 1$ versore (nu)

si dice DERIVATA DIREZIONALE DI f IN P_0 SECONDO v il limite

$$\lim_{t \to 0} \frac{f(P_0 + tv) - f(P_0)}{t} = \frac{\partial f}{\partial v}(P_0)$$



$g(t) = f(P_0 + vt) = f(x_0 + t v_1, y_0 + t v_2)$

$\lim_{t\to 0} \frac{g(t) - g(0)}{t}$ derivata di $g(t)$ rispetto a t in $t=0$.

Posso calcolare infinite derivate direzionali perchè infiniti sono i versori $v(\cos\theta, \operatorname{sen}\theta)$.

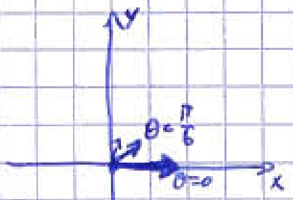
ESEMPIO

$f = \log(x + y^2 + 1)$ $\quad P_0 = (0,0)$ $\quad v = (\cos\theta, \operatorname{sen}\theta)$ $\quad \theta \in [0, 2\pi]$

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0} \frac{f((0,0) + t(\cos\theta, \operatorname{sen}\theta)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0 + t\cos\theta, 0 + t\operatorname{sen}\theta) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{\log(t\cos\theta + t^2\operatorname{sen}^2\theta + 1)}{t} \cdot \underbrace{\frac{t^2\operatorname{sen}^2\theta + t\cos\theta}{t^2\operatorname{sen}^2\theta + t\cos\theta}}_{\to 1} = \lim_{t\to 0}(t\operatorname{sen}^2\theta + \cos\theta) = \cos\theta$$ infatti sono ∞.

Se $\theta = 0$ $\quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial (1,0)}(0,0) = 1$



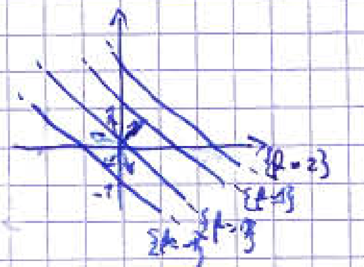
Se $\theta = \frac{\pi}{6}$ $\quad \frac{\partial f}{\partial (\frac{\sqrt{3}}{2}, \frac{1}{2})}(0,0) = \frac{\sqrt{3}}{2}$ ... sempre la 1ª componente.

molto inclinata su x
poco " su y

$f(x,y) = x + y$ $\quad \frac{\partial f}{\partial(\cos\theta, \operatorname{sen}\theta)}(0,0) = \lim_{t\to 0} \frac{f((0,0) + t(\cos\theta, \operatorname{sen}\theta)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t\cos\theta, t\operatorname{sen}\theta) - f(0,0)}{t} =$

$$= \lim_{t\to 0} \frac{t\cos\theta + t\operatorname{sen}\theta}{t} = \cos\theta + \operatorname{sen}\theta$$

$f(x,y) = k$



Se $v = (\cos\frac{7}{4}\pi, \operatorname{sen}\frac{7}{4}\pi) = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ $\quad \frac{\partial f}{\partial(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}(0,0) = 0.$

perchè v è parallela alla curva di livello

max di $\frac{\partial f}{\partial \theta}$

Se $v = (\cos\frac{\pi}{4}, \operatorname{sen}\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ $\quad \frac{\partial f}{\partial(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}(0,0) = 2\,\frac{\sqrt{2}}{2} = \sqrt{2}$

salgo rapidamente

$\frac{\partial f}{\partial(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})}(0,0) = -\sqrt{2}$ scendo rapidamente

minimo di $\frac{\partial f}{\partial \theta}$

DEF. $f: A \to \mathbb{R}$ A aperto $P_0 \in A$ se $\exists \lim\limits_{t \to 0} \dfrac{f(P_0 + t(1,0)) - f(P_0)}{t} = \dfrac{\partial f}{\partial x}(P_0)$

DERIVATA PARZIALE DI f FATTA RISPETTO A X IN P_0

mentre se $\exists \lim\limits_{t \to 0} \dfrac{f(P_0 + t(0,1)) - f(P_0)}{t} = \dfrac{\partial f}{\partial y}(P_0)$ DERIVATA PARZIALE DI f FATTA RISPETTO A Y IN P_0

$$f(x,y) \quad P_0 = (x_0, y_0) \quad \frac{\partial f}{\partial x}(P_0) = \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} \sim g(t) = f(P_0 + t(1,0)) = \frac{dg}{dt}(0) =$$

$$= \lim_{t \to 0} \frac{g(t) - g(0)}{t}$$

$$f(x,y) = \log(x + y^2 + 1) \quad \frac{\partial f}{\partial x}(1,1) = \lim_{t \to 0} \frac{\log((1+t) + 1^2 + 1) - \log(3)}{t} = \lim_{t \to 0} \frac{\log(3+t) - \log 3}{t} =$$

$$= \lim_{t \to 0} \frac{1}{3+t} = \frac{1}{3}$$

oppure penso alla y come un numero e penso alla x come variabile

$$f(x,y) = \log(x + y^2 + 1) \quad \frac{\partial f}{\partial x} = \frac{1}{x + y^2 + 1} \cdot \overset{\frac{\partial(x+y^2+1)}{\partial x}}{1} \Rightarrow \frac{\partial f}{\partial x}(1,1) = \frac{1}{3}$$

$$\frac{\partial f}{\partial y}(1,1) = \lim_{t \to 0} \frac{\log(1 + (1+t)^2 + 1) - \log(3)}{t} = \lim_{t \to 0} \frac{\log(3 + 2t + t^2) - \log(3)}{t} \overset{H}{=} \lim_{t \to 0} \frac{\frac{1}{3+2t+t^2} \cdot (2+2t) - 0}{1} = \frac{2}{3}$$

$$\text{oppure} \quad \frac{\partial f}{\partial y}(1,1) = \frac{1}{x + y^2 + 1} \cdot 2y = \frac{2}{1+1+1} = \frac{2}{3}$$

$$f(x,y) = x^2 + 3xy^3 + xy \qquad \frac{\partial f}{\partial x} = 2x + 3y^3 + y \qquad \frac{\partial f}{\partial y} = 9xy^2 + x$$

$$f(x,y) = 3x + 2xy + 6\,\text{sen}(xy) \qquad \frac{\partial f}{\partial x} = 3 + 2y + 6\cos(xy) \cdot y \qquad \frac{\partial f}{\partial y} = 2x + 6\cos(xy) \cdot x$$

DEF. (differenziabilità)

$f: A \to \mathbb{R}$ A aperto $P_0 = (x_0, y_0) \in A$

f è differenziale in P_0 se $\exists \overrightarrow{(a,b)} \in \mathbb{R}^2 : \lim\limits_{(x,y) \to (x_0,y_0)} \dfrac{f(x,y) - f(x_0,y_0) - \langle (a,b); (x-x_0, y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$

OSS

Questa definizione è la stessa data in $\mathbb{R}$.

L'applicazione $df(x_0,y_0) : (h,k) \to \langle (a,b); (h,k) \rangle$ si dice DIFFERENZIALE DI f IN (x_0,y_0)

Si può scrivere che $\exists (a,b) \in \mathbb{R}^2 : f(x,y) - f(x_0,y_0) - \langle (a,b); (x-x_0, y-y_0) \rangle \equiv \sigma(d((x,y);(x_0,y_0)$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\sigma\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

DEF.

Il piano $z = f(x_0,y_0) + \langle (a,b); (x-x_0, y-y_0) \rangle$, quando la funzione è differenziabile, si dice PIANO TANGENTE al grafico di f in $((x_0,y_0) f(x_0,y_0))$

OSS.

Se f è differenziabile $\Rightarrow f$ è continua e esiste il piano tangente

Se f è differenziabile $\Rightarrow f$ è derivabile secondo qualsiasi direzione $\exists \frac{\partial f}{\partial v}(x_0,y_0) \forall v$

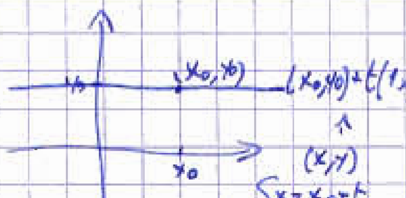
Se f è differenziabile in $(x_0,y_0) \Rightarrow (a,b) = \nabla f(x_0,y_0) \equiv$ GRADIENTE di f in $(x_0,y_0) \equiv$

$$\left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$$

Se f è differenziabile in $(x_0,y_0) \Rightarrow \forall v \quad v_1^2 + v_2^2 = 1 \quad \frac{\partial f}{\partial v}(x_0,y_0) = \langle \nabla f(x_0,y_0); v \rangle$

Se f è differenziabile $\exists (a,b) = \nabla f(x_0,y_0)$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - a(x-x_0) - b(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 \Rightarrow a = \frac{\partial f}{\partial x}(x_0,y_0)$$



$$\begin{cases} x = x_0 + t \\ y = y_0 \end{cases}$$

$$\lim_{t\to 0} \frac{f(x_0+t,y_0) - f(x_0,y_0) - a\,t - b\cdot 0}{\sqrt{t^2+0^2}} = \lim_{t\to 0} \frac{f(x_0+t,y_0) - f(x_0,y_0) - at}{|t|}$$

$$\begin{cases} x - x_0 = t \\ y - y_0 = 0 \end{cases}$$

DEFINIZIONE DI DERIVATA PARZIALE RISPETTO A X

Se $t>0$ $\lim_{t\to 0^+} \frac{f(x_0+t,y_0) - f(x_0,y_0)}{t} = a$ Se $t<0$ $\lim_{t\to 0^-} \frac{f(x_0+t,y_0) - f(x_0,y_0)}{\overset{+}{\ominus} t} = \overset{+}{\ominus} a$

$\Rightarrow a = \frac{df}{dx}(x_0,y_0)$ ▨ $\underline{(a,b) = \nabla f(x_0,y_0)} \leftarrow$ gradiente

TEOREMA DEL DIFFERENZIALE TOTALE

$f: A \to \mathbb{R}$ A aperto $(x_0,y_0) \in A$, se $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ esistono continue in un intor

di $(x_0,y_0) \Rightarrow f$ è differenziabile in (x_0,y_0)

ESEMPIO

$f(x,y)=x^2+2y^2x$; f è differenziabile $\forall (x,y)\in\mathbb{R}^2$? $\frac{\partial f}{\partial v}(1,2)=?$ $v=(\cos\theta, \operatorname{sen}\theta)$

il piano tangente nel punto $(1,2)=?$

$\frac{\partial f}{\partial x}=\underset{\text{continua su } \mathbb{R}^2}{2x+2y^2}$ $\frac{\partial f}{\partial y}=4xy$ continua su $\mathbb{R}^2$ $\overset{T(d.t.)}{\Rightarrow}$ f è differenziabile $\forall (x,y)\in\mathbb{R}^2$

$\nabla f=(2x+2y^2; 4xy)$ $\frac{\partial f}{\partial v}(1,2)=<\nabla f(1,2); \overset{v}{(\cos\theta, \operatorname{sen}\theta)}> = <(10,8); (\cos\theta,\operatorname{sen}\theta)> =$

$=10\cos\theta+8\operatorname{sen}\theta$

$z=f(1,2)+<\nabla f(1,2); (x-1; y-2)> = 9+<(10,8); (x-1; y-2)> = 9+10(x-1)+8(y-2)$

$z=10x+8y-17$

ESERCIZIO

f differenziabile in (x_0,y_0) allora $v=\frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$ VETTORE INDIVIDUATO DAL GRADIENTE $\frac{\partial f}{\partial v}(x_0,y_0)=\max\left\{\frac{\partial f}{\partial v}(x_0,y_0) : v=(\cos\theta,\operatorname{sen}\theta)\right\}$

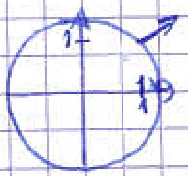
La direzione individuata dal gradiente è quella di massima pendenza

$$\frac{\partial f}{\partial(\cos\theta,\operatorname{sen}\theta)}(x_0,y_0)\leq\left|\frac{\partial f}{\partial(\cos\theta,\operatorname{sen}\theta)}(x_0,y_0)\right|\underset{f\text{ diff}}{\overset{\uparrow}{=}}\left|<\nabla f(x_0,y_0); (\cos\theta,\operatorname{sen}\theta)>\right|\leq\left|\nabla f(x_0,y_0)\cdot|(\cos\theta,\operatorname{sen}\theta)|\right|=$$

$=|\nabla f(x_0,y_0)|$ però $\frac{\partial f}{\partial v}(x_0,y_0)=<\nabla f(x_0,y_0); \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}>=\frac{|\nabla f(x_0,y_0)|}{|\nabla f(x_0,y_0)|}=\nabla f(x_0,y_0)$

ESEMPIO

$x^2+y^2=f$ $\nabla f=(2x,2y)$ $\nabla f(1,1)=(2,2)$



16/04/08

1) $\begin{cases}\exists (a,b)\in\mathbb{R}^2 \\ f(x,y)-f(x_0,y_0)=a(x-x_0)+b(x-x_0)+\sigma\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)\end{cases}$

$$0\leq|f(x,y)-f(x_0,y_0)|\leq\underbrace{|a||(x-x_0)|}_{\text{se }(x,y)\to(x_0,y_0)\text{ tende a }0}+\underbrace{|b||(x-x_0)|}_{0}+\underbrace{|0\sqrt{(x-x_0)^2+(y-y_0)^2}|}_{0}$$

$\Rightarrow |f(x,y)-f(x_0,y_0)|\xrightarrow[(x,y)\to(x_0,y_0)]{}0$ $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$ def. di continuità

3) Se f è continua $\not\Rightarrow$ f differenziabile

4) Se f è derivabile $\not\Rightarrow$ f differenziabile
$\forall$ direzione

5) Se f è derivabile $\forall$ direzione $\not\Rightarrow$ f continua

L'esistenza delle derivate direzionali non ci fa concludere niente.

GRADIENTE

f differenziabile in (x_0,y_0) e $\nabla f(x_0,y_0) \neq (0,0)$ allora $\frac{\partial f}{\partial v}(x_0,y_0) = \max \left\{ \frac{\partial f}{\partial v}(x_0,y_0);\ v = \begin{cases} \cos \\ \text{sen} \end{cases} \right.$

$$v = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

6) Se f differenziabile in (x_0,y_0) $\Rightarrow (a,b) = \nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0);\ \frac{\partial f}{\partial y}(x_0,y_0) \right)$

7) Se f differenziabile $\Rightarrow z = f(x_0,y_0) + \langle \nabla f(x_0,y_0), (x-x_0, y-y_0) \rangle$ è detto PIANO TANGENTE IN $(x_0, y_0, f(x_0,y_0))$.

ESEMPIO

$f(x,y) = (x \cancel{+} y)^2 \cancel{-2xy} + 3x - y - \cancel{x^2} - \cancel{y^2}$ Calcolare piano tg a f in $(3, \sqrt{3}, f(3,\sqrt{3}))$

$z = f(x,y) = 3x - y$ $\qquad \nabla f = (3, -1)$ sono continue $\Rightarrow$ vale T. diff. totale

$z = f(3,\sqrt{3}) + \langle \nabla f(3,\sqrt{3}), (x-3, y-\sqrt{3}) \rangle$

$z = 9 - \sqrt{3} + 3(x-3) - (y-\sqrt{3}) = \cancel{9} - \cancel{\sqrt{3}} + 3x \cancel{-9} - y + \cancel{\sqrt{3}} = 3x - y$ come ci aspett ~~x~~kè piano tg a un piano.

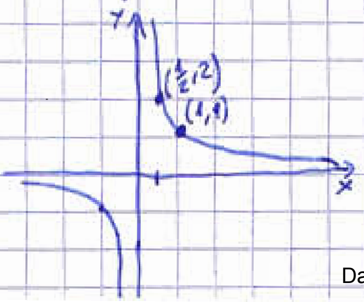
$f(x,y) = xy$ Calcolare piano tangente a $(1,1,f(1,1))$ e in $(\frac{1}{2}, 2, f(\frac{1}{2},2))$

$\nabla f = (y, x)$ cont. $z = f(1,1) + \langle \nabla f(1,1), [(x-1),(y-1)] \rangle = 1 + (x-1) + (y-1) = x + y - 1$

$z = f(\frac{1}{2}, 2) + \langle \nabla f(\frac{1}{2},2), [(x-\frac{1}{2}),(y-2)] \rangle = 1 + 2(x - \frac{1}{2}) + \frac{1}{2}(y-2) = 1 + 2x - 1 + \frac{1}{2}y - 1 =$
$= 2x + \frac{y}{2} - 1$

Intersecare il grafico di $f(x,y)$ con l'insieme di livello $z=1$

$$\begin{cases} z = xy \\ z = 1 \end{cases}$$



Calcolo la retta tangente nei due punti.

$y=\frac{1}{x}$ retta tangente in $(1,1)$ $y'=-\frac{1}{x^2}$ $y-1=-1(x-1)$ $y=-x+2$

retta tangente in $(\frac{1}{2},2)$ $y'=-\frac{1}{x^2}$ $y-2=-4(x-\frac{1}{2})$ $y=-4x+4$

Piano tangente in $(1,1,f(1,1))$; $z=x+y-1$ $\begin{cases} z=x+y-1 \\ z=1 \end{cases}$ $\begin{cases} z=x+y-1 \\ z=1 \end{cases}$ $\begin{cases} x+y-2=0 \\ z=1 \end{cases}$

Retta tangente in $(1,1)$: $y+x-2=0$ ←

⇒ intersecando il piano tangente in $(1,1,f(1,1))$ con $z=1$ trovo la retta tangente

OSS.

Se f è differenziabile in (x_0,y_0) e $\nabla f(x_0,y_0)\neq(0,0)$, allora

$z=f(x_0,y_0)+<\nabla f(x_0,y_0),(x-x_0,y-y_0)>$ è l'equazione del piano tangente

$<\nabla f(x_0,y_0),((x-x_0),(y-y_0))>=0$ è l'equazione della retta tangente a $\{f(x,y)=f(x_0,y_0)\}$ in $\{x_0,y_0\}$

Calcolo la retta tangente in $(\frac{1}{2},2)$

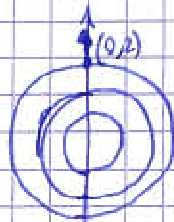
$<\nabla f(\frac{1}{2},2),((x-\frac{1}{2}),(y-2))>=0$ $2\left(x-\frac{1}{2}\right)+\frac{1}{2}(y-2)=0$ $2x-1+\frac{1}{2}y-1=0$

$4x-4+y=0$ $y=-4x+4$

Il gradiente è ortogonale alla curva di livello!

La direzione di massima pendenza è ortogonale alla curva di livello ed è la direzione per andar su il più rapidamente possibile.

$f(x,y)=x^2+y^2$



$\nabla f=(2x,2y)$

$\frac{\nabla f(0,2)}{|\nabla f(0,2)|}=\frac{(0,4)}{4}=(0,1)$ direzione già individuata

DEF.

$f:A\to\mathbb{R}$ A aperto di $\mathbb{R}^2$ $(x_0,y_0)\in A$ e definisco $\forall\nu,\mu\in\mathbb{R}^2$ $|\mu|=|\nu|=1$

DERIVATA SECONDA

$$\frac{\partial^2 f}{\partial\nu\partial\mu}(x_0,y_0)=\frac{\partial}{\partial\nu}\left(\frac{\partial f}{\partial\mu}\right)(x_0,y_0)$$

$$\frac{\partial^2 f}{\partial\mu\partial\nu}(x_0,y_0)=\frac{\partial}{\partial\mu}\left(\frac{\partial f}{\partial\nu}\right)(x_0,y_0)$$

ESEMPIO

$$f(x,y) = x^3 + 3xy^2 + 3y \qquad \frac{\partial f}{\partial x} = 3x^2 + 3y^2$$

$$\frac{\partial f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial x \partial x} = 6y$$

$$\frac{\partial f}{\partial y} = 6xy + 3$$

$$\frac{\partial f}{\partial y^2} = 6x \qquad \frac{\partial f}{\partial x \partial y} = 6y$$

$=$

DEF. MATRICE HESSIANA DI f

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \nabla\left(\frac{\partial f}{\partial x}\right) \\ \nabla\left(\frac{\partial f}{\partial y}\right) \end{pmatrix}$$

Nel caso precedente

$$H_f(x,y) = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

ESEMPIO

$$f(x,y) = xe^y + y^3 x^2 \qquad \frac{\partial f}{\partial x} = e^y + 2xy^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 \qquad \frac{\partial^2 f}{\partial y \partial x} = e^y + 6xy^2$$

$$\frac{\partial f}{\partial y} = xe^y + 3x^2y^2$$

$$\frac{\partial^2 f}{\partial y^2} = xe^y + 6x^2y \qquad \frac{\partial^2 f}{\partial x \partial y} = e^y + 6xy^2$$

$=$

$$H_{f(x,y)} = \begin{pmatrix} 2y^3 & e^y + 6xy^2 \\ e^y + 6xy^2 & xe^y + 6x^2y \end{pmatrix}$$

$$f(x,y) = x^2 + |y|$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x\right) = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \overset{?}{=} \qquad (x_0, y_0) = (0,0)$$

TEOREMA DI SCHWARZ

Data $f: A \to \mathbb{R}^2$ A aperto $(x_0, y_0) \in A$ f continua, $\frac{\partial f}{\partial x}$ e $\frac{\partial f}{\partial y}$ continue, $\frac{\partial^2 f}{\partial x \partial y}$ e $\frac{\partial^2 f}{\partial y \partial x}$ continue in un intorno di (x_0, y_0), allora

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0)$$

DEF. Data $f: A \to \mathbb{R}$ A aperto in $\mathbb{R}^2$ $(x_0,y_0) \in A$ f differenziabile in (x_0,y_0)

(x_0,y_0) è PUNTO STAZIONARIO $\iff \nabla f(x_0,y_0) = (0,0)$ cioè $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

OSS. Se $\nabla f(x_0,y_0) = (0,0)$ e f diff. $\Rightarrow z = f(x_0,y_0)$ è il piano tangente

ESEMPIO

$f(x,y) = x^2+y^2$ $\nabla f = (2x,2y)$ piano tangente in $(0,0,0)$ è $\underset{\text{piano } /\!/ xy}{z = f(0,0) + \langle \nabla f(0,0),(x,y) \rangle = 0}$

Tra i punti stazionari ci sono i punti di MASSIMO e MINIMO.

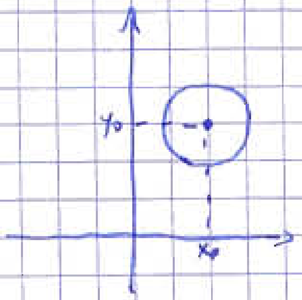
TEOREMA

Se $f: A \to \mathbb{R}$ differenziabile in A, A aperto $\subseteq \mathbb{R}^2$ e

(x_0,y_0) punto di minimo o massimo relativo interno

$\Rightarrow \nabla f(x_0,y_0) = (0,0)$

DIM.



Suppongo (x_0,y_0) minimo relativo

$f(x,y) \geq f(x_0,y_0)$ $\forall (x,y) \in B(x_0,y_0;R)$

$g(x) = f(x,y_0)$ restrizione

$g(x) \geq g(x_0)$ $\forall x \in]x_0-R, x_0+R[$ g diff. in x_0, g ha minimo in $x_0 \Rightarrow g'(x_0) = 0$.

$\Rightarrow \frac{\partial f}{\partial x}(x_0,y_0) = 0$ $\underset{g(x) = f(x_0,y_0)}{\text{perchè}}$ Analogamente per y $\therefore$ $\frac{\partial f}{\partial y}(x_0,y_0) = 0 \Rightarrow \nabla f = 0$

ESEMPI

$f(x,y) = x^2+y^2$ $\nabla f(x,y) = (2x,2y) \to (0,0)$ $\overset{\text{unico}}{\vee}$ punto stazionario $z=0$ è piano tg

$g(x,y) = 4-x^2-y^2$ $\nabla g(x,y) = (-2x,-2y) \to (0,0)$ $\overset{\text{unico}}{\vee}$ punto stazionario $z=4$ è piano tg

$h(x,y) = x^2-y^2$ $\nabla h(x,y) = (2x,-2y) \to (0,0)$ $\overset{\text{unico}}{\vee}$ punto stazionario $z=0$ è piano tg

Nel caso di $f(x)$, $(0,0)$ è un punto di minimo relativo interno perchè

$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ha autovalori >0

Nel caso di $g(x)$, $(0,0)$ è un punto di massimo relativo interno perchè

$H_g = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ ha autovalori <0

Nel caso di $h(x)$, $(0,0)$ non è massimo nè minimo, ma PUNTO DI SELLA,

perchè $H_h = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ ha un autovalore >0 e uno <0.

21/04/08

λ è autovalore della matrice A se $\exists \vec{b} \neq 0 : \; A\vec{b} = \lambda \vec{b} = \lambda Id \vec{B}$

$(A - \lambda Id)\vec{b} = 0. \quad \det(A - \lambda Id) = 0$

$$A = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) , & \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) , & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} \cdot a_{21} = 0 \qquad \lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0$

$\lambda^2 - \lambda \, Tr A + \det A = 0 \qquad A$ simmetrica $\Rightarrow \exists \lambda_1, \lambda_2 \in \mathbb{R}$

$\det A =$ prodotto radici

$Tr A =$ somma radici

LEMMA

Se $\det A > 0$ e $Tr A > 0 \Rightarrow \lambda_1, \lambda_2 > 0$

Se $\det A > 0$ e $Tr A < 0 \Rightarrow \lambda_1, \lambda_2 < 0$

Se $\det A < 0 \Longrightarrow \lambda_1 < 0 < \lambda_2$

In realtà basta sapere il segno di a_{11} invece che la $Tr A$.

TEOREMA (condizione sufficiente)

Data $f : A \to \mathbb{R}$, A aperto, $(x_0, y_0) \in A$ e tale che

- $f \in C^2(A)$
- $\nabla f(x_0, y_0) = (0,0)$

allora:

1) se $\det H_f(x_0,y_0) > 0$ e $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) > 0$ allora (x_0, y_0) è un punto di MINIMO RELAT

2) se $\det H_f(x_0,y_0) > 0$ e $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) < 0$ allora (x_0,y_0) è un punto di MASSIMO RELATIV

3) se $\det H_f(x_0,y_0) < 0$ allora (x_0,y_0) è un punto di sella

Se $\det H_f(x_0,y_0) = 0$, questo teorema non si applica.

ESEMPIO

$f(x,y) = 2x^3 + y^3 - 3x^2 - 3y$ Determinare i punti stazionari e studiarne la natura; determinare $\sup_{\mathbb{R}^2} f$ e $\inf_{\mathbb{R}^2} f$. $\longrightarrow f(0,y) = y^3 - 3y \xrightarrow[y \to \pm\infty]{} \pm\infty$

$+\infty$ è $\sup_{\mathbb{R}^2} f$

$-\infty$ è $\inf_{\mathbb{R}^2} f$.

$\nabla f (6x^2 - 6x, 3y^2 - 3)$

$$\nabla f = (0,0) \begin{cases} 6x^2 - 6x = 0 \\ 3y^2 - 3 = 0 \end{cases} \quad \begin{cases} x(x-1) = 0 \\ y^2 = 1 \end{cases} \quad \begin{cases} x = 0 \lor x = 1 \\ y = 1 \lor y = -1 \end{cases}$$

$P_1 = (0,1) \quad P_2 = (0,-1)$

$P_3 = (1,1) \quad P_4 = (1,-1)$

$H_f(x,y)=\begin{pmatrix}12x-6 & 0\\ 0 & 6y\end{pmatrix}$ $H_f(P_1)=\begin{pmatrix}-6 & 0\\ 0 & 6\end{pmatrix}$ P_1 punto di sella

$H_f(P_2)=\begin{pmatrix}-6 & 0\\ 0 & -6\end{pmatrix}$ $\frac{\partial^2 f}{\partial x^2}(0,-1)=-6<0$ P_2 punto di massimo relativo

$H_f(P_3)=\begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix}$ $\frac{\partial^2 f}{\partial x^2}(1,1)=6>0$ P_3 punto di minimo relativo

$H_f(P_4)=\begin{pmatrix}6 & 0\\ 0 & -6\end{pmatrix}$ P_4 punto di sella

$f(x,y)=4x^2+y^4-2xy^2+6x-3y^2+3$ Determinare i punti stazionari e loro natura

$\nabla f(x,y)=(8x-2y^2+6,\ 4y^3-4xy-6y)=(0,0)$ $\begin{cases}8x-2y^2+6=0\\ 4y^3-4xy-6y=0\end{cases}$

$\begin{cases}4x=y^2-3\\ 2y^3-2xy-3y=0\end{cases}$ $\begin{cases}2x=\frac{1}{2}(y^2-3)\\ 2y^3-\frac{1}{2}(y^2-3)\cdot y-3y=0\end{cases}$

$\begin{cases}\dots\\ 2y^3-\frac{1}{2}y^3+\frac{3}{2}y-3y=0\end{cases}$

$4y^3-y^3+3y-6y=0$

$3y^3-3y=0$

$y(y^2-1)=0$

$\begin{cases}y=0\\ x=-\frac{3}{4}\end{cases}$ $A=\left(-\frac{3}{4},0\right)$

$\begin{cases}y=1\\ x=-\frac{1}{2}\end{cases}$ $B=\left(-\frac{1}{2},1\right)$ $\begin{cases}y=-1\\ x=-\frac{1}{2}\end{cases}$ $C=\left(-\frac{1}{2},-1\right)$

$H_f(x,y)=\begin{pmatrix}8 & -4y\\ -4y & 12y^2-4x-6\end{pmatrix}$ $H_f(A)=\begin{pmatrix}8 & 0\\ 0 & -3\end{pmatrix}$ A) è un punto di sella

$H_f(B)=\begin{pmatrix}8 & -4\\ -4 & 12+2-6\end{pmatrix}$ $\det(H_f(B))=8\cdot 8-16>0$ $\frac{\partial^2 f}{\partial x^2}(B)>0$ B è punto di minimo

$H_f(C)=\begin{pmatrix}8 & 4\\ 4 & 12+2-6\end{pmatrix}$ $\det(H_f(C))=8\cdot 8-16>0$ $\frac{\partial^2 f}{\partial x^2}(C)>0$ C è punto di minimo.

TEOREMA (Weierstrass)

Data $f: C \to \mathbb{R}$ $\quad C \subseteq \mathbb{R}^2$

- C chiuso e limitato
- f continua

$\Rightarrow \exists \min_C f = f(x_m, y_m) \quad (x_m, y_m) \in C$

$\max_C f = f(X_n, Y_n) \quad (X_n, Y_n) \in C$

ESEMPIO $\quad$ A: palla con frontiera

$f: \overline{B(0,0;1)} \to \mathbb{R}$

$f(x,y) = \dfrac{1+x}{1+x^2+y^2}$ Determinate max e min di f su A (se esistono)

$\overline{B(0,0;1)} = \underbrace{\{(x,y) : x^2+y^2<1\}}_{B(0,0;1)} \cup \underbrace{\{(x,y) : x^2+y^2=1\}}_{\partial B(0,0;1)}$

$\overline{B(0,0;1)}$ è chiuso (contiene la sua frontiera) e limitato.

$f(x,y)$ è continua $\Rightarrow$ vale il T di Weierstrass $\Rightarrow \exists$ max e min su $\overline{B(0,0;1)}$

1) Cerco i punti stazionari di f che stanno dentro $B(0,0;1)$ studiandone la natura.

2) Cerco il max e il min di f su $\partial B(0,0;1)$

3) $\max_{\overline{C}} f = \max\{\max_{\partial C} f \,;\, \max_C f\}$ $\quad C = B(0,0;1)$

$\min_{\overline{C}} f = \min\{\min_{\partial C} f \,;\, \min_C f\}$

$$\nabla f = \left(\frac{1+x^2+y^2-(1+x)\cdot 2x}{(1+x^2+y^2)^2} ; \frac{-2y(1+x)}{(1+x^2+y^2)^2}\right) = \left(\frac{y^2-x^2-2x+1}{(1+x^2+y^2)^2} ; \frac{-2y(1+x)}{(1+x^2+y^2)^2}\right)$$

$$\nabla f(x,y) = 0 \quad \begin{cases} y^2-x^2-2x+1=0 \\ -2y(1+x)=0 \end{cases} \quad \begin{cases} \ldots \\ y(1+x)=0 \end{cases} \quad \begin{cases} y=0 \\ x_{1,2} = -1 \pm \sqrt{1+1} \end{cases} \quad \begin{matrix} A = (-1+\sqrt{2}, 0) \\ \underbrace{B = (-1-\sqrt{2}, 0)}_{\text{fuori da } B(0,0;1)} \end{matrix}$$

$\hookrightarrow x = -1 \notin B(0,0;1)$

$H_f(A) = \ldots$

Studio f su $\partial B(0,0;1) = \{(\cos t, \operatorname{sen} t) : t \in [0, 2\pi]\}$

Studio $g(t) = f(\varphi(t)) = \dfrac{1+\cos t}{1+\cos^2 t + \operatorname{sen}^2 t} = \dfrac{1}{2}(1+\cos t) \quad t \in [0, 2\pi] \quad g'(t) = \dfrac{1}{2}(-\operatorname{sen} t) = 0$

$\operatorname{sen} t = 0 \iff t = 0 \,,\, t = \pi \qquad g''(t) = -\dfrac{1}{2}\cos t \qquad g''(0) = -\dfrac{1}{2} < 0 \to g(0) = 1$ max relati

$g''(\pi) = \dfrac{1}{2} > 0 \to g(\pi) = 0$ min relativo

$g(0) = g(2\pi) = 1$

$\max_{\partial B(0,0;1)} f = 1 = g(0) = f(1,0) \qquad \min_{\partial B(0,0;1)} f = 0 = g(\pi) = f(-1,0)$

$$f(-1+\sqrt{2},0) = \frac{1-1+\sqrt{2}}{1+(-1+\sqrt{2})^2} = \frac{\sqrt{2}}{4-2\sqrt{2}} = \frac{1}{2(\sqrt{2}-1)} \overset{?}{>} 1 \qquad 1 \overset{?}{>} 2\sqrt{2}-2 \qquad 3 \overset{?}{>} 2\sqrt{2} \qquad 9 \overset{?}{>} 8 \text{ si}$$

$$\Rightarrow \max_{\overline{B(0,0;1)}} f = \max\left\{\max_{\partial B(0,0;1)} f, \max_{B(0,0;1)} f\right\} = \max\left\{1; \frac{1}{2(\sqrt{2}-1)}\right\} = \frac{1}{2(\sqrt{2}-1)} = f(-1+\sqrt{2},0)$$

$$\min_{\overline{B(0,0;1)}} f = \min_{\partial B(0,0;1)} f = f(-1,0) = 0 \qquad f\left(\overline{B(0,0;1)}\right) = \left[0, \frac{1}{2\sqrt{2}-2}\right]$$

↳ perché all'interno non ci sono minimi

28/04/08

EQUAZIONI DIFFERENZIALI

$y'(x) = \cos x$ chi è $y(x)$? $y(x) = \operatorname{sen} x + c$ $c \in \mathbb{R}$

- L'incognita $y(x)$ è una funzione
- In generale, si trovano infinite soluzioni

$y'(x) = y(x)$ come fare?

*) Una soluzione è $y(x) = 0$.

*) Se $y(x) \neq 0$ in un intorno di x_0 $\Rightarrow \frac{y'(x)}{y(x)} = 1$ $\int \frac{y'(x)}{y(x)} dx = \int 1\, dx$ $\int \frac{y'(x)}{y(x)} dx = x + c$ $z = y(x)$ $dz = y'(x) \cdot dx$

$\int \frac{1}{z} \cdot dz = x + c$ $\log|z| = x + c$ $\log|y(x)| = x + c$

$|y(x)| = e^{x+c} = C \cdot e^x$

OSS.

Nessuna soluzione attraversa $y(x) = 0$ (la soluzione costante). Il segno è sempre ben determinato

$\Rightarrow y(x) = k \cdot e^x$ $k \in \mathbb{R}$ sono tutte le soluzioni

PROBLEMA DI CAUCHY

Data $f: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, determinare le soluzioni di $\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$ e stabilire se è unica.

ESEMPIO

$\begin{cases} y'(x) = \cos x \\ y(0) = 0 \end{cases}$ $y(x) = \operatorname{sen} x$ $\begin{cases} y'(x) = \cos x \\ y(0) = 1 \end{cases}$ $y(x) = \operatorname{sen} x + c$ $y(0) = 0 + c = 1$ $C = 1$

$y(x) = \operatorname{sen} x + 1$

$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ $y(x) = ke^x$ $y(0) = ke^0 = 1$ $\Rightarrow k = 1$ $y(x) = e^x$

$\begin{cases} y' = y \\ y(0) = \pi \end{cases}$ $y(x) = ke^x$ $y(0) = ke^0 = \pi$ $k = \pi$ $y(x) = \pi e^x$

$\begin{cases} y'(x) = y \\ y(1) = -1 \end{cases}$ $y(x) = ke^x$ $y(1) = ke^1 = -1$ $k = -\frac{1}{e}$ $y(x) = -\frac{1}{e}e^x = -e^{x-1}$

TEOREMA (CAUCHY-LIPSCHITZ)

Data $f: \Omega \in \mathbb{R}^{n+1} \to \mathbb{R}^n$ f continua con derivate continue ($\underline{f \in C^1(\Omega)}$)
Dato $\underline{(x_0, y_0) \in \Omega}$ con $x_0 \in \mathbb{R}$ e $y_0 \in \mathbb{R}^n$ allora esiste $\delta > 0$ ed esiste un $y(x) \in C^1(]x_0 - \delta; x_0 + \delta[; \mathbb{R}^n)$ soluzione del problema di Cauchy

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

COROLLARIO

Nelle ipotesi del teorema di Cauchy-Lipschitz due soluzioni diverse <u>non</u> hanno punti in comune (i due grafici non si intersecano)

DIM.

Siano $y(x) = y_1(x)$ e $y(x) = y_2(x)$ due soluzioni di $y' = f(x,y)$ $f \in C^1(\Omega)$ e $y_1(x) \neq y_2(x)$. Se per assurdo $\exists\, x_0 \in dom(y_1) \cap dom(y_2)$ tale che $y_1(x_0) = y_2(x_0) = y_0$ allora essendo la soluzione di $\begin{cases} y'(x) = f(x,y) \\ y(x_0) = y_0 \end{cases}$ ottengo un assurdo.

<u>ESEMPIO</u>

$y' = \underbrace{2x}_{a(x)}\underbrace{y^2}_{b(y)}$ è di classe C^1 e quindi vale il T. di Cauchy-Lipschitz. Osserviamo che $b(0) = 0$, allora $y(x) = 0$ è soluzione costante.

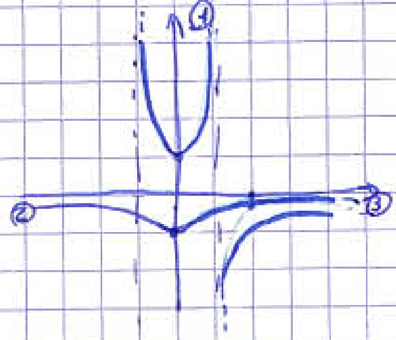
Dunque una qualsiasi soluzione diversa da $y = 0$ sarà sempre > 0
$< 0 \Rightarrow \frac{y'(x)}{y^2(x)} = 2x$ $\int \frac{y'(x)}{y^2(x)} dx = x^2 + c$ $\left(\int \frac{dz}{z^2}\right)_{z = y(x)} = x^2 + c$ $-\frac{1}{y(x)} = x^2 + c$

$y(x) = -\frac{1}{x^2 + c}$

Impongo il passaggio per $(0,1)$ $y(0)=-\frac{1}{0+c}=1$ $c=-1$ $y(x)=-\frac{1}{x^2-1}$ ①

Impongo " " " $(0,-1)$ $y(0)=-\frac{1}{0+c}=-1$ $c=1$ $y(x)=-\frac{1}{x^2+1}$ ②

Impongo " " " $\left(2,-\frac{1}{3}\right)$. $y(2)=-\frac{1}{4+c}=-\frac{1}{3}$ $4+c=3$ $c=-1$ ③



La soluzione del problema di Cauchy con dato iniziale $y(x_0)=y_0$ viene considerato nel più grande intervallo contenente x_0 in cui $y(x)\in C^1$.

$$\begin{cases} y'=f(x,y) \\ y(x_0)=y_0 \end{cases}$$ $y(x)$ è soluzione se

- $y'(x)=f(x,y(x))$ $\forall x\in]a,b[$
- $y(x_0)=y_0$ con $(x_0,y_0)\in dom(f)$
- $y\in C^1(]a,b[)$ con $]a,b[\ni x_0$

EQUAZIONI DIFFERENZIALI LINEARI

1° ORDINE

$\underbrace{y'(x)=a(x)\,y+b(x)}_{(*)}$ con $a(x), b(x) :]a,b[\to\mathbb{R}$ continue

Stiamo cercando una funzione incognita e compare la derivata prima (equazione differenziale) e solo la derivata prima (di 1° ordine).

OSS.

Se y_1 e y_2 sono soluzioni di (*) allora $c_1y_1+c_2y_2$ è ancora soluzione.

DIM.

$z=c_1y_1+c_2y_2$ $z'=c_1y_1'+c_2y_2'=(a(x)\cdot y_1)\cdot c_1+c_2(a(x)\cdot y_2)=a(x)(c_1y_1+c_2y_2)=$

$=a(x)\cdot z$ $\Rightarrow z$ è soluzione ■

$y'(x)=a(x)\cdot y(x)+b(x)$ $y'(x)-a(x)\cdot y(x)=b(x)$ TECNICA DEL FATTORE INTEGRANT

Detta $A(x)$ una primitiva di $a(x)$ (ovvero $A'(x)=a(x)$)

$y'(x)e^{-A(x)}-e^{-A(x)}a(x)\cdot y(x)=e^{-A(x)}\cdot b(x)$ $\left(y(x)\cdot e^{-A(x)}\right)'=$ 1° membro

$$\left(y(x)\, e^{-A(x)}\right)' = e^{-A(x)}\, b(x) \qquad y(x)\cdot e^{-A(x)} = \int e^{-A(x)}\cdot b(x)\, dx$$

$$\boxed{y(x) = e^{A(x)} \left[\int e^{-A(x)} \cdot b(x)\right]}$$

ESEMPIO

$$\begin{cases} y' = (\tan x)\cdot y + \operatorname{sen}(2x) \\ y(0) = 0 \end{cases} \qquad a(x) = \tan(x) \quad \operatorname{dom}(a(x)) = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$A(x) = \int \frac{\operatorname{sen} x}{\cos x}\, dx = -\ln|\cos x| + c \qquad A(x) = -\ln|\cos x| \quad \overset{c=0}{\downarrow}$$

+ sempl

In $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ $\cos x > 0$ $A(x) = -\ln \cos x$.

$$\int b(x)\cdot e^{-A(x)}\, dx = \int 2\operatorname{sen} x \cos x \cdot e^{\ln \cos x}\, dx = \int 2 \operatorname{sen} x \cos^2 x\, dx = -2\,\frac{\cos^3 x}{3} + c$$

$$-\frac{2}{3}\cos^3 x + c = \int e^{-A(x)} \cdot b(x)$$

$$y(x) = e^{-\ln \cos x} \cdot \left(-\frac{2}{3}\cos^3 x + c\right) = \frac{1}{\cos x}\cdot\left(-\frac{2}{3}\cos^3 x + c\right) \quad \text{cerco } y(0) = 0$$

$$\frac{1}{\cos 0}\cdot\left(-\frac{2}{3}\cos^3 0 + c\right) = 0 \qquad -\frac{2}{3} + c = 0 \qquad c = \frac{2}{3} \qquad y(x) = -\frac{2}{3}\cos^2 x + \frac{2}{3}\cdot\frac{1}{\cos x}$$

05/05/08

METODO VARIAZIONE COSTANTI ARBITRARIE $y' = a(x)y + b(x)$

1) Considero l'equazione omogenea associata $y' = a(x)\cdot y$. Questa ha come integrale generale $y_0(x) = c\cdot e^{A(x)}$ $A'(x) = a(x)$

$$y'(x) = \left(c\, e^{A(x)}\right)' = c\cdot e^{A(x)}\cdot a(x) = a(x)\cdot (c\, y(x))$$

2) Cerchiamo $y(x)$ soluzione di $y' = a(x)y + b(x)$. Cerco una soluzione del tipo $v(x) = c(x)\cdot e^{A(x)}$. Impongo $v(x)$ soluzione di $y' = a(x)y + b(x)$, cioè $v'(x) = a(x)v(x) + b(x)$

$$c'(x)\, e^{A(x)} + \cancel{c(x)\cdot a(x)\, e^{A(x)}} = \cancel{a(x)\, c(x)\, e^{A(x)}} + b(x)$$

$$\Rightarrow c'(x) = b(x)\, e^{-A(x)} \qquad k + c(x) = \int b(x)\cdot e^{-A(x)}\, dx$$

$$y_p(x) = e^{A(x)}\cdot c(x) = e^{A(x)}\cdot\int b(x)\cdot e^{-A(x)}\, dx$$

SOLUZ.
PARTICOLARE

3) Per l'osservazione che seguirà, l'integrale generale è

$$y(x) = \underbrace{c \cdot e^{A(x)}}_{\text{INTEGRALE GEN. OMOGENEA}} + \underbrace{e^{A(x)} \int b(x) e^{-A(x)} dx}_{\text{SOLUZIONE PARTICOLARE}}$$

OSS.

Se $y(x)$ è soluzione di $y'(x) = a(x)y + b(x)$, così come $z(x)$, allora esiste $c \in \mathbb{R}$: $y(x) - z(x) = c \cdot e^{A(x)}$

DIM.

$$(y-z)' = y' - z' = a(x) \cdot y(x) + \cancel{b(x)} - \left(a(x) \cdot z(x) + \cancel{b(x)}\right) = a(x) \cdot \left[y(x) - z(x)\right]$$. SOLUZ. EQUAZ. OMOGENEA

ESEMPIO

$$y' = 3x^2 y + (1-3x^2) \cdot e^x$$

1) $y' = 3x^2 y$ $\quad a(x) = 3x^2 \quad \overset{A = \int a(x)}{A(x)} = x^3 \quad y_0(x) = c \cdot e^{x^3}$ integrale generale omogenea associata $c \in \mathbb{R}$

2) $v(x) = c(x) \cdot e^{x^3}$ impongo $v'(x) = 3x^2 v + (1-3x^2) e^x$ risolvo l'equazione

$c'(x) e^{x^3} + \cancel{c(x) \cdot 3x^2 \cdot e^{x^3}} = \cancel{3x^2 \cdot c(x) \cdot e^{x^3}} + (1-3x^2) e^x \qquad c'(x) = (1-3x^2) e^x \cdot e^{-x^3}$

$c(x) = \int (1-3x^2) e^{x-x^3} dx = e^{x-x^3} + k$ selezione $k=0$

$y_p(x) = \underset{c(x)}{e^{x-x^3}} \cdot e^{x^3} = e^x$

3) $y(x) = y_0(x) + y_p(x) = c \cdot e^{x^3} + e^x \quad c \in \mathbb{R}$ ▨

Nel caso di equazioni NON lineari il metodo non funziona

$y' = 3x^2 y^2 + e^x(1-3x^2)$

1. $y' = 3x^2 y^2 \qquad \frac{y'}{y^2} = 3x^2 \qquad \int \frac{y'(x)}{y^2(x)} dx = x^3 + c \qquad -\frac{1}{y(x)} = x^3 + c \qquad y_0(x) = -\frac{1}{x^3+c} \quad c \in \mathbb{R}$

($y=0$ sol. cost)

2. $v(x) = -\frac{1}{x^3+c(x)}$ impongo $v'(x) = 3x^2 v^2(x) + (1-3x^2) e^x$

$$+\frac{(3x^2+c'(x))}{(x^3+c(x))^2} = 3x^2 \cdot \frac{1}{(x^3+c(x))^2} + e^x(1-3x^2) \qquad \frac{c'(x)}{\left[x^3+c(x)\right]^2} = e^x(1-3x^2)$$

Non integrabile direttamente, riottengo un'equazione differenziale in $c(x)$.

Questo perché NON È LINEARE!!

EQUAZIONI DIFFERENZIALI LINEARI DI ORDINE n A COEFFICIENTI COSTANTI

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x)$$

$a_0, \dots, a_n \in \mathbb{R}$ $\quad f \in C(]a,b[)$

OSS.

Se $n=1$ allora trovo $a_1 y' + a_0 y = f(x)$ e risolvo l'omogenea associata $a_1 y' + a_0 y = 0$

$y' = -\frac{a_0}{a_1} y$ $\quad y_0 = k e^{-\frac{a_0}{a_1}x}$ ... (caso precedente) $\lambda = -\frac{a_0}{a_1}$ è la radice di $a_1 \lambda + a_0 = 0$

EQUAZIONE OMOGENEA ASSOCIATA

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$ $\quad$ Nel caso $n=1$ si è trovata una soluzione del tipo $e^{\lambda x}$

Cerchiamo le soluzioni dell'omogenea associata del tipo $e^{\lambda x}$

$$a_n \frac{d^n(e^{\lambda x})}{dx^n} + \dots + a_1 \cdot (e^{\lambda x})' + a_0 (e^{\lambda x}) = 0 \qquad a_n \lambda^n \cdot e^{\lambda x} + a_{n-1} \lambda^{n-1} \cdot e^{\lambda x} + \dots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

$\underset{\neq 0}{e^{\lambda x}} (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0) = 0$ $\quad$ EQUAZIONE CARATTERISTICA in campo complesso di ordine n.

TEOREMA FONDAMENTALE DELL'ALGEBRA

Data un'equazione algebrica di grado n, questa possiede in $\mathbb{C}$ n soluzioni, eventualmente non distinte. ↑ non dice come si calcolano

Dunque, dall'equazione caratteristica trovo $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ eventualmente non distinte che sono radici.

Noti $\lambda_1, \dots, \lambda_n$ sono note $e^{\lambda_1 x} \dots e^{\lambda_n x}$ (soluzioni equazione omogenea)

TEOREMA

$S = \{$soluzioni equazione omogenea ordine $n\}$ è uno spazio vettoriale di ordine n

- 1° CASO: SOLUZIONI REALI E DISTINTE

Se $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ allora $e^{\lambda_1 x} \dots e^{\lambda_n x}$ sono soluzioni dell'equazione omogenea; inoltre

$y_0(x) = c_1 e^{\lambda_1 x} + \dots + c_n e^{\lambda_n x}$ $\quad c_1, \dots, c_n \in \mathbb{R}$

ESEMPIO

$y''' - y' = 0$ $\quad$ L'equazione caratteristica $\lambda^3 - \lambda = 0$ $\quad \lambda(\lambda^2 - 1) = 0$ $\quad \lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$

$\Rightarrow \underbrace{e^{0x}}_{y_1}, \underbrace{e^{1x}}_{y_2}, \underbrace{e^{-1x}}_{y_3} \Rightarrow y_0 = c_1 \cdot y_1(x) + c_2 y_2(x) + c_3 y_3(x)$ $\quad c_1, c_2, c_3 \in \mathbb{R}$

•• 2° CASO : SOLUZIONI REALI COINCIDENTI

esempio

$$y''' = 0 \xrightarrow[3\ \text{volte}]{\text{integro}} y_0(x) = c_1 x^2 + c_2 x + c_3$$

Se ho una radice reale con molteplicità k (ovvero $(\lambda-\lambda_*)^k$ è fattore dell'equazione caratteristica).

$$\lambda_* \longrightarrow \begin{cases} e^{\lambda_* \cdot x} \\ x e^{\lambda_* \cdot x} \\ x^2 e^{\lambda_* \cdot x} \\ \vdots \\ x^{k-1} e^{\lambda_* \cdot x} \end{cases}$$

Nel caso precedente $\lambda_* = 0$

$e^{\lambda_* \cdot x} = e^{0 \cdot x} = 1$ $\quad \overset{\downarrow}{\lambda^3} = 0$ molt. 3

$y_1 = e^{0 \cdot x} = 1 \quad y_2 = x \cdot e^{0x} = x \quad y_3 = x^2$

ESEMPIO

$y'' - 2y' + y = 0 \quad \lambda^2 - 2\lambda + 1 = 0 \quad (\lambda-1)^2 = 0 \quad \lambda = 1$ con molteplicità 2

$y_1 = e^x \quad y_2 = x e^x$

Verifico che e^x è soluzione: $e^x - 2e^x + e^x = 0$ ok

Verifico che xe^x è soluzione: $y' = e^x + xe^x = e^x(x+1)$

$y'' = e^x + e^x + xe^x = e^x(2+x)$

$e^x(2+x) - 2e^x(x+1) + e^x \cdot x = 0 \quad \cancel{2e^x} + \cancel{xe^x} - 2\cancel{x}e^x - \cancel{2e^x} + \cancel{x}e^x = 0$ ok

$y_0 = c_1 \cdot e^x + c_2 x e^x$

∴ 3° CASO : SOLUZIONI COMPLESSE DISTINTE

Se $\lambda_1^{\in \mathbb{C}}$ è radice complessa semplice (molteplicità 1) allora $e^{\lambda_1 x}$ è soluzione dell'omogenea. $e^{\lambda_1 x} = y_1$ è a valori in $\mathbb{C}$.

OSS.

Se $\lambda_* \in \mathbb{C}$ è radice di $a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$ con $a_n, \dots, a_0 \in \mathbb{R}$ allora $\overline{\lambda_*}$ è anche esse radice.

DIM.

Se $a_n \lambda_*^n + \dots + a_1 \lambda_*^1 + a_0 = 0$ anche il suo coniugato è 0. Inoltre

$\overline{a_n \lambda_*^n + \dots + a_1 \lambda + a_0} = 0$ e $a_n \cdot (\overline{\lambda_*})^n + \dots + a_1 \overline{\lambda} + a_0 = 0$ ■

Se $\lambda_* = \alpha + i\beta$ è radice equazione caratteristica allora $\overline{\lambda_*} = \alpha - i\beta$ è radice.

$$\begin{matrix}\lambda_* \\ \overline{\lambda_*}\end{matrix} \longrightarrow \begin{matrix} e^{\lambda_* x} = e^{\alpha x}\cdot e^{i\beta x} \\ e^{\overline{\lambda_*} x} = e^{\alpha x}\cdot e^{-i\beta x}\end{matrix} \longrightarrow \frac{e^{\alpha x}\cdot e^{i\beta x}+e^{\alpha x}\cdot e^{-i\beta x}}{2}$$

SEMISOMMA $e^{\alpha x}\cdot\frac{e^{i\beta x}+e^{-i\beta x}}{2}$

$$\frac{e^{\alpha x}\cdot e^{i\beta x}-e^{\alpha x}\cdot e^{-i\beta x}}{2i}$$

SEMIDIFFERENZA $e^{\alpha x}\cdot\frac{e^{i\beta x}-e^{-i\beta x}}{2i}$

$\Rightarrow \begin{matrix} e^{\alpha x}\cdot\cos\beta x \\ e^{\alpha x}\cdot\text{sen}\,\beta x\end{matrix}$ SOLUZIONI REALI

Se $\lambda\in\mathbb{R}$, $\beta(x)=0 \Rightarrow e^{\alpha x}$.

ESEMPIO (MOLLA)

$$x''(t) = -\frac{k}{m}x(t) \qquad \lambda^2+\frac{k}{m}=0 \qquad \lambda_1 = i\sqrt{\frac{k}{m}} \qquad \lambda_2 = -i\sqrt{\frac{k}{m}}$$

$$x_1 = \cos\left(\sqrt{\frac{k}{m}}\cdot t\right) \qquad x_2 = \text{sen}\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

$$x_0(t) = C_1\cdot\cos\left(\sqrt{\frac{k}{m}}\cdot t\right)+C_2\cdot\text{sen}\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

:: 4° CASO: SOLUZIONI COMPLESSE CON MOLTEPLICITÀ > 1

Se $\lambda_*\in\mathbb{C}$ ha molteplicità $k>1$, allora $\exists\ \overline{\lambda_*}\in\mathbb{C}$ con molteplicità $k>1$ radice. Allora:

$$\begin{matrix}\lambda_* = \alpha+i\beta \\ \overline{\lambda_*} = \alpha-i\beta\end{matrix} \longrightarrow \begin{matrix} 1)\begin{cases} e^{\alpha x}\cdot\cos\beta x \\ e^{\alpha x}\cdot\text{sen}\,\beta x\end{cases} \\ 2)\begin{cases} xe^{\alpha x}\cdot\cos\beta x \\ xe^{\alpha x}\cdot\text{sen}\,\beta x\end{cases} \\ \vdots \\ k\begin{cases} x^{k-1}e^{\alpha x}\cdot\cos\beta x \\ x^{k-1}e^{\alpha x}\cdot\text{sen}\,\beta x\end{cases}\end{matrix}$$

ESEMPIO

$$y^{(IV)}+2y''+y=0 \qquad \lambda^4+2\lambda^2+1=0 \text{ EQ. CARATT.}$$

$$(\lambda^2+1)^2=0 \qquad \begin{matrix}\lambda_1 = i \\ \lambda_2=-i\end{matrix}\ \text{molt.}=2 \longrightarrow \begin{matrix}\begin{cases}\text{sen}x = y_1 \\ \cos x = y_2\end{cases} \\ \begin{cases} x\,\text{sen}x = y_3 \\ x\cos x = y_4\end{cases}\end{matrix}$$

$$y_0(x) = C_1\cos x + C_2\,\text{sen}x + C_3 x\,\text{sen}x + C_4 x\cos x \qquad C_1, C_2, C_3, C_4 \in\mathbb{R}$$

07/05/08

$a_n y^{(n)}+\ldots+a_1 y'+a_0 y = f(x)$

(1) Cerco l'integrale generale dell'omogenea $a_n y^{(n)}+\ldots+a_0 y=0$ che chiamo $y_0(x)$

(2) Cerco una soluzione $y_p(x)$ della completa

(3) $y_g(x) = y_0(x)+y_p(x)$ integrale generale della completa

(1) si fa con il polinomio caratteristico...

(2) di seguito dei casi:

$f(x)$ polinomio in x, polinomio in sen$x$, cos$x$, polinomio in e^{kx}.

Trovo una soluzione dello stesso tipo

ESEMPIO

$y'' - y = e^{2x}$ 1) omog. ass. $y''-y=0$ $\lambda^2-1=0$ $\lambda_1=1$ $y_1=e^x$

$\lambda_2=-1$ $y_2=e^{-x}$

$y_0(x)=c_1\cdot e^x+c_2\cdot e^{-x}$ $c_1,c_2\in\mathbb{R}$

2) $f(x)=e^{2x}$ Cerco una soluzione del tipo $v(x)=k\cdot e^{2x}$. Impongo $v''-v=e^{2}$

$4ke^{2x}-ke^{2x}=e^{2x}$ $3k=1$ $k=\frac{1}{3}$ $y_p=\frac{1}{3}e^{2x}$

3) $y_g(x)=c_1\cdot e^x+c_2\cdot e^{-x}+\frac{1}{3}e^{2x}$ $c_1,c_2\in\mathbb{R}$

* * *

$y'''-y=x^2$ 1) $y'''-y=0$ $\lambda^3-1=0$ $\lambda_1=+1$ $y_1(x)=e^x$

$\lambda_2=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ $y_2(x)=e^{-\frac{x}{2}}\,\text{sen}\left(\frac{\sqrt{3}}{2}x\right)$

$\lambda_3=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ $y_3(x)=e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$

$y_0(x)=c_1e^x+c_2e^{-\frac{x}{2}}\,\text{sen}\left(\frac{\sqrt{3}}{2}x\right)+c_3e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$ $c_1,c_2,c_3\in\mathbb{R}$

2) $f(x)=x^2$ polinomio 2° grado. Cerco soluzione $v(x)=ax^2+bx+c$. Impongo $v'''-v=x^2$

↑ generico polinomio grado 2

$0-ax^2-bx-c=x^2$ $\begin{cases}a=-1\\b=0\\c=0\end{cases}$ $\Rightarrow$ $y_p(x)=-x^2$

3) $y_g(x)=c_1e^x+e^{-\frac{x}{2}}\left(c_2\,\text{sen}\left(\frac{\sqrt{3}}{2}x\right)+c_3\cos\left(\frac{\sqrt{3}}{2}x\right)\right)-x^2.$

* * *

$y''-y'=\cos x$ 1) $y''-y'=0$ $\lambda^2-\lambda=0$ $\lambda(\lambda-1)=0$ $\lambda_1=0$ $y_1=e^{0x}=1$

$\lambda_2=1$ $y_2=e^x$

$y_0(x)=c_1+c_2e^x$ $c_1,c_2\in\mathbb{R}$

2) $f(x)=\cos x$ polinomio in sen e cos. Cerco soluzione $v(x)=a\cos x+b\,\text{sen}x$

$v'(x)=-a\,\text{sen}x+b\cos x$ $v''(x)=-a\cos x-b\,\text{sen}x$

$-a\cos x-b\,\text{sen}x-(-a\,\text{sen}x+b\cos x)=\cos x$

$\text{sen}x(a-b)+\cos x(-a-b)=\cos x$ $\begin{cases}a-b=0\\-a-b=1\end{cases}$ $\begin{cases}a=b\\-b-b=1\end{cases}$ $\begin{cases}b=-\frac{1}{2}\\a=-\frac{1}{2}\end{cases}$

$y_p(x)=-\frac{1}{2}(\text{sen}x+\cos x)$

3) $y_g(x)=c_1+c_2e^x-\frac{1}{2}(\text{sen}x+\cos x)$ $c_1,c_2\in\mathbb{R}$

$y'' + y = \cos(\beta x)$ Cerco $y_g(x)$ al variare di $\beta \geq 0$

1) $y''+y=0$ (equazione della molla quando $\frac{k}{m}=1$) $\lambda^2+1=0$ $\lambda_1 = i$ $\lambda_2=-i$

$y_1 = \cos x$ $y_2 = \operatorname{sen} x$ $y_0(x) = C_1 \cos x + C_2 \operatorname{sen} x$ $C_1, C_2 \in \mathbb{R}$

$$\boxed{\begin{array}{l}\underline{\text{OSS}}\\ y_0(x) \text{ è LIMITATA } \forall x \in \mathbb{R}\end{array}}$$

2) $f(x) = \cos(\beta x)$ (rappresenta una forza esterna variabile con intensità uguale a 1). Distinguo 3 casi

(i) $\beta = 0$ $f(x) = 1$ $y_p(x) = 1$

(ii) $\beta \neq 1$ $f(x) =$ polinomio in $\operatorname{sen}\beta x, \cos\beta x$. Cerco soluzione $v(x) = a\cos\beta x + b \operatorname{sen}\beta x$

$v'(x) = -a\beta \operatorname{sen}\beta x + b\beta\cos\beta x$ $v''(x) = -a\beta^2\cos\beta x - b\beta^2 \operatorname{sen}\beta x$

$-a\beta^2\cos\beta x - b\beta^2\operatorname{sen}\beta x + a\cos\beta x + b\operatorname{sen}\beta x = \cos\beta x$ $\begin{cases}-a\beta^2+a=1\\ -b\beta^2+b=0\end{cases} \xrightarrow{\beta\neq1} \begin{cases}a(1-\beta^2)=1\\ b(1-\beta^2)=0\end{cases}$

$\begin{cases} a = \frac{1}{1-\beta^2} \\ b = 0\end{cases}$ $y_p = \dfrac{1}{1-\beta^2}\cos(\beta x)$

(iii) $\beta = 1$ (caso risonante) $y''+y=\cos x$ ma $a\cos x + b\operatorname{sen} x$ è soluzione dell'omogenea

$\Rightarrow$ moltiplico per x la $v(x)$ $\Rightarrow$ $v(x) = (a\cos x + b \operatorname{sen} x)\cdot x$

$v'(x) = a\cos x + b\operatorname{sen} x - ax\operatorname{sen} x + bx\cos x$ $v''(x) = -a\operatorname{sen} x + b\cos x - a\operatorname{sen} x - ax\cos x + b\cos x - b\operatorname{sen} x$

$= [-2a\operatorname{sen} x + 2b\cos x - \underline{x[a\cos x + b\operatorname{sen} x]}] + \underline{ax\cos x + bx\operatorname{sen} x} = \cos x$

$\begin{cases}-2a=0\\ 2b=1\end{cases}$ $\begin{cases}a=0\\ b=\frac{1}{2}\end{cases}$ $y_p(x) = \left(0\cdot\cos x + \frac{1}{2}\operatorname{sen} x\right)\cdot x = \frac{1}{2}x\operatorname{sen} x$

3)

(i) $y_g(x) = C_1\cos x + C_2 \operatorname{sen} x + 1$ $\beta = 0$

(ii) $y_g(x) = C_1 \cos x + C_2\operatorname{sen} x + \dfrac{1}{1-\beta^2}\cos\beta x$ $\beta \neq 1$

} FUNZIONI LIMITATE

(iii) $y_g(x) = C_1\cos x + C_2 \operatorname{sen} x + \frac{1}{2}x\operatorname{sen} x$ ← FUNZIONE ILLIMITATA

$y''' - y'' = x^2$ 1) $\lambda^3 - \lambda^2 = 0$ $\lambda^2(\lambda-1)=0$ $\lambda_1 = 1$ $\lambda_2 = 0$ 2 volte $< \begin{array}{l} y_1 = e^{0x} = 1 \\ y_2 = e^{0x}\cdot x = x\end{array}$

$\lambda_1 = 1 \to y_3 = e^x$

$y_0(x) = C_1 + C_2 x + C_3 e^x$ $C_1, C_2, C_3 \in \mathbb{R}$

2) $f(x) = x^2$ polinomio di grado 2 $v(x) = ax^2 + bx + c$ $v'(x) = 2ax + b$ $v''(x) = 2a$ $v'''(x) =$

$-2a = x^2$ non ha soluzione perchè a dipende da x.

⇒ moltiplico per x $v(x)=ax^3+bx^2+cx$, ma ancora una volta non va bene

⇒ " ancora per x $v(x)=ax^4+bx^3+cx^2$ $v'(x)=4ax^3+3bx^2+2cx$

$v''(x)=12ax^2+6bx+2c$ $v'''(x)=24ax+6b$

$24ax+6b-12ax^2-6bx-2c=x^2$ $\begin{cases}-12a=1\\24a-6b=0\\6b-2c=0\end{cases}$ $\begin{cases}a=-1/12\\b=-1/3\\c=-1\end{cases}$

$y_p(x)=-\frac{1}{12}x^4-\frac{1}{3}x^3-x^2$

3) $y_g(x)=C_1+C_2x+C_3e^x-\frac{1}{12}x^4-\frac{1}{3}x^3-x^2$ $C_1,C_2,C_3\in\mathbb{R}$

— . —— .

$y''-y'=xe^x$ 1) $\lambda^2-\lambda=0$ $\lambda(\lambda-1)=0$ $\lambda_1=0$ $\lambda_2=1$ $y_0=1$ $y_1=e^x$

$y_0(x)=c_1+c_2e^x$

è soluzione dell'omogenea

2) $f(x)=$ (polinomio 1° grado)$\cdot(e^x)$. Cerco $v(x)=(ax+b)\cdot e^x=axe^x+be^x$ molt. per x

$v(x)=ax^2e^x+bxe^x$ $v'(x)=2axe^x+ax^2e^x+be^x+bxe^x$

$v''(x)=2ae^x+2axe^x+2axe^x+ax^2e^x+be^x+be^x+bxe^x$

$2ae^x+4axe^x+\cancel{ax^2e^x}+2be^x+\cancel{bxe^x}-2axe^x-\cancel{ax^2e^x}-be^x-\cancel{bxe^x}=xe^x$

$2a+2ax+b=x$ $\begin{cases}2a+b=0\\2a=1\end{cases}$ $\begin{cases}a=\frac{1}{2}\\b=-1\end{cases}$ $y_p(x)=\left(\frac{1}{2}x-1\right)\cdot e^x\cdot x$

3) $y_g(x)=c_1+c_2e^x+xe^x\left(\frac{1}{2}x-1\right)$

OSS

Un'altra soluzione particolare può essere $\left(\frac{x^2}{2}-x\right)e^x+\underbrace{\overset{\text{viene assorbito da } c_2}{\pi e^x}\overset{\text{viene assorbito da } c_1}{-4}}_{\text{soluzione dell'omogenea}}$

— . — . —

$y''-y=\operatorname{sen}x+x^2$

Uso il principio di sovrapposizione degli "effetti" /soluzioni $\left\langle\begin{matrix}y''-y=\operatorname{sen}x\\ \oplus\\ y''-y=x^2\end{matrix}\right.$

Funziona perchè sono lineari

1) $\lambda^2-1=0$ $\lambda^2=1$ $\lambda_1=+1$ $\lambda_2=-1$ $y_0(x)=c_1e^x+c_2e^{-x}$ $c_1,c_2\in\mathbb{R}$

2) Cerco sol. part. di $y''-y=\operatorname{sen}x$ $v(x)=a\operatorname{sen}x+b\cos x$ senx non è sol. di $y_0(x)$

$v'(x)=a\cos x-b\operatorname{sen}x$ $v''(x)=-a\operatorname{sen}x-b\cos x$ $-a\operatorname{sen}x-b\cos x-a\operatorname{sen}x-b\cos x=\operatorname{sen}x$

$\begin{cases}-2a=1\\-2b=0\end{cases}$ $\begin{cases}a=-1/2\\b=0\end{cases}$ $y_{p1}(x)=-\frac{1}{2}\operatorname{sen}x$

Cerco $y_{p_{11}}(x)$ di $y''-y=x^2$ $\quad v(x)=ax^2+bx+c$ non sol. omogenea

$v'(x)=2ax+b \quad v''(x)=2a \quad 2a-ax^2-bx-c=x^2$ $\quad \begin{cases}-a=1\\-b=0\\2a-c=0\end{cases} \quad \begin{cases}a=-1\\b=0\\c=-2\end{cases}$

$y_{p_{11}}(x)=-x^2-2$

3) $y_g(x)=c_1e^x+c_2e^{-x}-\frac{1}{2}\operatorname{sen}x-x^2-2$. $\quad c_1,c_2\in\mathbb{R}$

—— . —— . —— . —— . ——

$\begin{cases}y''-y=\operatorname{sen}x+x^2\\ y(0)=0\end{cases}$ ottengo ∞^1 soluzioni (senza $y(0)=0$ avrei ∞^2 soluzioni)

$\begin{cases}y(0)=c_1+c_2-2=0 \Rightarrow c_1=2-c_2\\ y(x)=(2-c_2)e^x+c_2e^{-x}-\frac{1}{2}\operatorname{sen}x-x^2-2\end{cases}$ $\quad y_g'(x)=c_1e^x-c_2e^{-x}-\frac{1}{2}\cos x-2x$

$y'(0)=c_1-c_2-\frac{1}{2}$

$\begin{cases}y(0)=0\\y'(0)=0\end{cases} \quad \begin{cases}c_1+c_2=2\\c_1-c_2=\frac{1}{2}\end{cases} \quad \begin{cases}c_1=\frac{5}{4}\\c_2=\frac{3}{4}\end{cases}$ $\quad \exists!$ soluzione $y(x)=\frac{5}{4}e^x+\frac{3}{4}e^{-x}-\frac{1}{2}\operatorname{sen}x-x^2-2$

ESERCIZI

(1) $y'''-y=x$ con condizioni $\begin{cases}y(0)=0\\y'(0)=0\\y''(0)=1\end{cases}$ $\quad$ (2) $y'''-y'=0$ $\begin{cases}y(0)=0\\y'(0)=1\\y''(0)=2\end{cases}$ $\exists!$ soluzione

La garanzia della soluzione unica ce l'ho solo se do condizioni in un solo punto partendo dalla funzione nel punto.

$\begin{cases}y'=x\\y(0)=1\end{cases}$ $\quad y_g=\frac{x^2}{2}+c \quad y_g(0)=1 \Rightarrow \frac{0}{2}+c=1 \Rightarrow c=1 \quad y=\frac{x^2}{2}+1.$

$\begin{cases}y'=x\\y'(0)=3\end{cases}$ $\quad y_g=\frac{x^2}{2}+c \quad y_g'=x \quad y_g'(0)=0=3$ IMPOS. perchè ho dato condizioni sulla derivata di ordine massimo

19/05/08

$\mathcal{M}$ insiemi misurabili in $\mathbb{R}^2$

1) R rettangolo $\Rightarrow R\in\mathcal{M}$

2) $E\in\mathcal{M} \Rightarrow (x_0,y_0)+E=\{(x,y):(x-x_0,y-y_0)\in E\}\in\mathcal{M}$

3) $T:\mathbb{R}^2\to\mathbb{R}^2$ è una rotazione, $E\in\mathcal{M} \Rightarrow T(E)\in\mathcal{M}$

4) $E_i\in\mathcal{M}$ $i=1\ldots n$, $E_i\cap E_j=\phi$ per $i\neq j \Rightarrow \bigcup_{i=1}^{n}E_i\in\mathcal{M}$ $\quad m\left(\bigcup_{i=1}^{n}E_i\right)=\sum_{i=1}^{n}\overset{\substack{\text{misura}\\ \downarrow}}{m}(E_i)$

5) $E, F \in \mathcal{M}$ $E \subset F \Rightarrow m(E) \leq m(F)$

Fare una teoria della misura significa:

- identificare gli insiemi misurabili
- associare ad $E \in \mathcal{M}$ $m(E) \in \mathbb{R}$

DEF. PLURIRETTANGOLO

$P = \bigcup_{i=1}^{n} R_i$ dove R_i è un rettangolo e $R_i \cap R_J \subset (\partial R_i \cap \partial R_J)$ $i \neq J$, cioè possono avere in comune al massimo un lato.

$m(P) = \sum_{i=1}^{n} m(R_i)$

DEF.

$E \subseteq \mathbb{R}^2$ limitato si dice MISURABILE se $\forall \varepsilon > 0$ $\exists P_\varepsilon^- \subset E$ plurirettangolo e $\exists P_\varepsilon^+ \supset E$ plurirettangolo tale che $m(P_\varepsilon^+) - m(P_\varepsilon^-) < \varepsilon$.

OSS.

Se $A \in \mathcal{M}$ e $m(A) = 0$ e $E \subset A \Rightarrow E \in \mathcal{M}$, $m(E) = 0$.

ESEMPIO

$E = \{(x,y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ con f continua su a, b, allora E è misurabile e $m(E) = \int_a^b f(x)\,dx$

TEOREMA

E, F misurabile $(E, F \in \mathcal{M})$ allora

1) $E \cup F$, $E \cap F$, $E \setminus F$ sono misurabili
2) $m(E \cup F) \leq m(E) + m(F)$
3) se $E \cap F = \phi \Rightarrow m(E \cup F) = m(E) + m(F)$
4) $\bar{E} = E \cup \partial E$ misurabile, $m(\bar{E}) = m(E) = m(E \setminus \partial E)$

TEOREMA

Se $E \subseteq \mathbb{R}^2$ limitato, allora E misurabile $\Leftrightarrow$ ∂E misurabile e $m(\partial E) = 0$

ESEMPIO

Se E è limitato $(E \subseteq \mathbb{R}^2)$ e $\partial E = \varphi([a,b])$, con φ curva regolare C^1 a tratti allora $m(\partial E) = m(\varphi([a,b])) = 0$

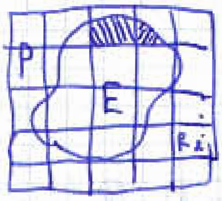
Sia E misurabile, $f : E \Rightarrow \mathbb{R}$ limitata.

DEF.

$\Delta = \{E_i,\ i=1\dots k\}$ suddivisione di E se

1) $E_i \in \mathcal{M}$ 2) $\bigcup_{i=1}^{k} E_i = E$ 3) $E_i \cap E_j \subseteq \partial E_i \cap \partial E_j$ $i \neq j$ intersezione con misura nulla

ESEMPIO



P plurirettangolo $P \supset E$

$P = \bigcup_{i=1}^{k} R_i$

$\Rightarrow \{E \cap R_i,\ i=1\dots k\}$ suddivisione "

$$s(f,\Delta) = \sum_{i=1}^{k} \left(\inf_{E_i} f(x,y)\right) \cdot m(E_i) \qquad S(f,\Delta) = \sum_{i=1}^{k} \left(\sup_{E_i} f(x,y)\right) \cdot m(E_i)$$

DEF.

$f: E \to \mathbb{R}$ con E misurabile e f limitata, è integrabile se

$$\sup_{\Delta} s(f,\Delta) = s(f) = S(f) = \inf_{\Delta} S(f,\Delta) = \int_E f(x,y)\,dx\,dy$$

OSS.

$$m(E) = \int_E 1\,dx\,dy$$

TEOREMA

A,B misurabile, $f,g: \Omega \to \mathbb{R}$ integrabile con $A,B \subset \Omega$

1) $\int_A (f+g)\,dx\,dy = \int_A f\,dx\,dy + \int_A g\,dx\,dy$

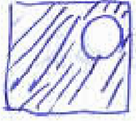
2) $\int_A \lambda f(x,y)\,dx\,dy = \lambda \int_A f(x,y)\,dx\,dy$

3) $f \leq g$ $(x,y) \in A \Rightarrow \int_A f(x,y)\,dx\,dy \leq \int_A g(x,y)\,dx\,dy$

4) $m(A) \cdot \inf_A f(x,y) \leq \int_A f\,dx\,dy \leq m(A) \cdot \sup_A f(x,y)$

5) $m(A \cap B) = 0$ allora $\int_{A \cup B} f\,dx\,dy = \int_A f\,dx\,dy + \int_B f\,dx\,dy$

6) $A\subset B \Rightarrow \int_{B\setminus A} f\,dx\,dy = \int_{B} f\,dx\,dy - \int_{A} f\,dx\,dy$



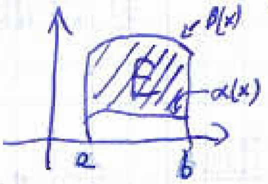
$\int$ quadrato $-\int$ cerchio

TEOREMA

$E\subset\mathbb{R}^2$ misurabile, $f:E\to\mathbb{R}$ continua, allora f è integrabile.

$E=\{(x,y): a\le x\le b \quad \alpha(x)\le y\le \beta(x)\}$

$F=\{(x,y): c\le y\le d \quad \gamma(y)\le x\le \delta(y)\}$



DEF.

E si dice DOMINIO NORMALE rispetto all'asse X se $\exists\, \alpha,\beta:[a,b]\to\mathbb{R}$ con $\alpha(x)\le\beta(x)$ $x\in[a,b]$.

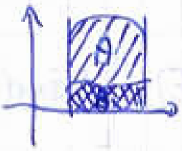
F si dice DOMINIO NORMALE rispetto all'asse Y se $\exists\, \gamma,\delta:[c,d]\to\mathbb{R}$ con $\gamma(y)\le\delta(y)$ $y\in[c,d]$.

OSS.

E dominio normale rispetto asse x $[y]$ allora E misurabile.

D.IM.

$E=\{(x,y): a\le x\le b \quad 0\le y\le\beta(x)\}\setminus\{(x,y): a\le x\le b \quad 0\le y\le\alpha(x)\} = A\setminus B$



A e B sono misurabili $\Rightarrow E=A\setminus B$ misurabile

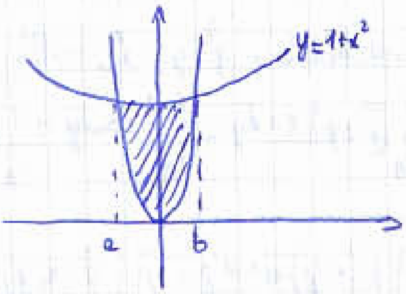
ESEMPI

1) insieme delimitato da $y=10x^2$ e $y=1+x^2$

$\alpha(x)=10x^2 \quad \beta(x)=1+x^2$

$$\begin{cases} y=10x^2 \\ y=1+x^2 \end{cases} \quad 10x^2=1+x^2 \quad x^2=\frac{1}{9} \quad x=\pm\frac{1}{3}$$

$a=-\frac{1}{3} \quad b=\frac{1}{3}$ $\qquad E=\left\{(x,y): -\frac{1}{3}\le x\le\frac{1}{3},\ 10x^2\le y\le 1+x^2\right\}$

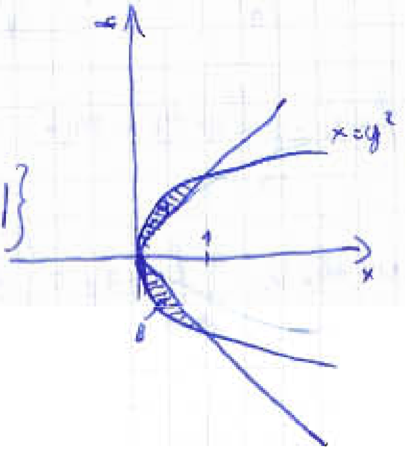


2) F dominio limitato da $x=y^2$ e $x=|y|$

$\gamma(y)=y^2 \quad \delta(y)=|y|$

$y^2=|y| \quad c=-1$
$\quad d=1$

$F=\{(x,y): -1\le y\le 1,\ y^2\le x\le|y|\}$



$A=\{(x,y)\ \ 0\le x\le 1,\ x\le y\le\sqrt{x}\}$

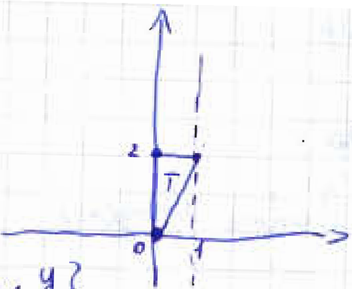
$B=\{(x,y)\ \ 0\le x\le 1,\ -\sqrt{x}\le y\le -x\}$

3) T triangolo di vertici $(0,0), (0,2), (1,2)$

Trovo i 3 tratti:

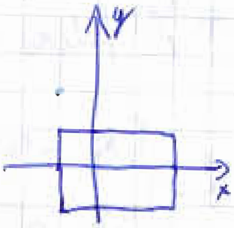
$x=0$, $y=2$, $y=2x$

$T=\{(x,y): 0\leq x\leq 1,\ 2x\leq y\leq 2\} = \{(x,y): 0\leq y\leq 2,\ 0\leq x\leq \frac{y}{2}\}$



4) Rettangolo $[-1,2]\times[-1,1]$

$P=\{(x,y)\ -1\leq x\leq 2,\ -1\leq y\leq 1\}$



TEOREMA DI RIDUZIONE

$f: \Omega \rightarrow \mathbb{R}$ continua

1) se $A=\{(X,y): a\leq x\leq b,\ \alpha(x)\leq y\leq \beta(x)\}\subseteq \Omega$ allora $U(x)=\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$ è continua e

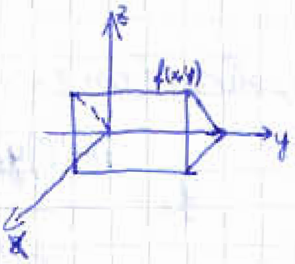
$$\int_A f(x,y)\,dx\,dy = \int_a^b U(x)\,dx$$

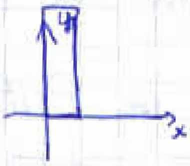
2) se $B=\{(x,y): c\leq y\leq d,\ \gamma(y)\leq x\leq \delta(y)\}\subseteq\Omega$ allora $V(y)=\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx$ è continua e

$$\int_B f(x,y)\,dx\,dy = \int_c^d V(y)\,dy$$

ESEMPIO

$f(x,y)=3x \qquad R=[0,1]\times[0,3]$





$R=\{(x,y): 0\leq y\leq 3,\ 0\leq x\leq 1\}$

$V(y)=\int_0^1 3x\,dx = = \left[\frac{3}{2}x^2\right]_0^1 = \frac{3}{2} \qquad \int_R f(x,y)\,dx\,dy = \int_0^3 \frac{3}{2}\,dy = \left[\frac{3}{2}y\right]_0^3 = \frac{9}{2}.$

$U(x)=\int_0^3 3x\,dy = 9x \qquad \int_R f(x,y)\,dx\,dy = \int_0^1 9x\,dx = \frac{9}{2}.$

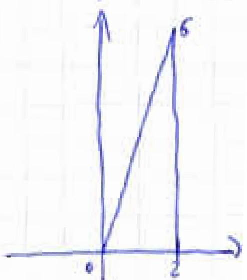
$\int_A f(x,y)\,dx\,dy = \int_0^2 dx\left(\int_0^{3x} f(x,y)\,dy\right)$

1) chi è A

2) invertire l'ordine d'integrazione

1) $0\leq x\leq 2 \quad 0\leq y\leq 3x \quad A=\{(x,y): 0\leq x\leq 2,\ 0\leq y\leq 3x\}$

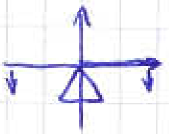
2) $B=\{(x,y): 0\leq y\leq 6,\ \frac{y}{3}\leq x\leq 2\} \quad \int_B f\,dx\,dy = \int_0^6 dy\left(\int_{\frac{y}{3}}^{2} f(x,y)\,dx\right)$



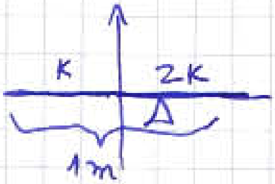
se $f(x,y) = xy$

22/05/08

BARICENTRO



bastone omogeneo $\int_{\varphi} x\, ds = 0$



bastone non omogeneo → Il fulcro sarà spostato verso destra

coordinate del baricentro $\dfrac{\int_{\varphi} x \cdot f \cdot ds}{\int_{\varphi} f\, ds}$

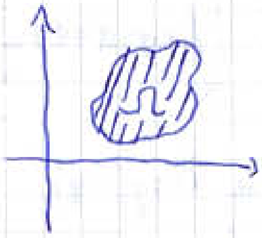
$$f = \begin{cases} K & -1 \le x \le 0 \\ 2K & 0 < x \le 1 \end{cases}$$

BARICENTRO → punto in cui le forze per il braccio si equilibriano

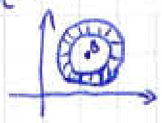
DEF.

$\Omega \subseteq \mathbb{R}^2$ misurabile, $\rho : \Omega \to \mathbb{R}$ continua (densità di massa)

$$x_B = \frac{\int_{\Omega} x\, \rho(x,y)\, dx\, dy}{\int_{\Omega} \rho(x,y)\, dx\, dy} \quad \text{MASSA TOTALE} \qquad y_B = \frac{\int_{\Omega} y\, \rho(x,y)\, dx\, dy}{\int_{\Omega} \rho(x,y)\, dx\, dy}$$



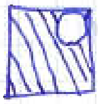
1) Non è detto che $(x_B, y_B) \in \Omega$



2) $\Omega = E \cup F$ con $E \cap F = \phi$, allora (con ρ = costante)

$$x_{\Omega} = \frac{m(E)\cdot x_E + m(F)\cdot x_F}{m(\Omega)} \qquad y_{\Omega} = \frac{m(E) y_E + m(F) y_F}{m(\Omega)} \qquad m(\Omega) = m(E) + m(F)$$

3) $\Omega = E \setminus F$ con ρ = costante



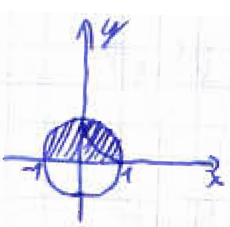
$$x_{\Omega} = \frac{m(E)\cdot x_E - m(F) x_F}{m(E) - m(F)} \qquad y_{\Omega} = \frac{m(E)\cdot y_E - m(F)\cdot y_F}{m(E) - m(F)}$$

se $E = F$, il baricentro sparisce

4) Ω possiede un'asse di simmetria allora $(x_{\Omega}, y_{\Omega}) \in$ asse (se ρ = costante)

5) Ω convesso e ρ = costante allora $(x_{\Omega}, y_{\Omega}) \in \Omega$.

Calcolare il baricentro di $\Omega = \{(x,y);\ x^2+y^2 \leq 1\ ,\ y \geq 0\}$



Dominio normale rispetto a x : $\Omega = \{(x,y): -1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}\}$

" " " " y : $\Omega = \{(x,y): 0 \leq y \leq 1,\ -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\}$

ρ = costante

Cerco la massa : $m(\Omega) = \int_\Omega 1\, dxdy = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy = \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$

$$\int \sqrt{1-x^2}\, dx \quad \underset{d\,\mathrm{sen}t = \cos t\, dt}{x = \mathrm{sen}t} \quad \int \sqrt{1-\mathrm{sen}^2 t} \cdot \cos t\, dt = \int |\cos t| \cdot \cos t\, dt \underset{\substack{\text{prendo} \\ [-\frac{\pi}{2}, \frac{\pi}{2}]}}{=} \int \cos^2 t\, dt = \ldots\ldots$$

$$\int_\Omega x\, dxdy = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} x\, dy = \int_{-1}^{1} dx \cdot \underset{\text{FUNZIONE}}{x\sqrt{1-x^2}} = -\frac{1}{2}\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} \cdot -2x\, dx = \left[\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}} \cdot \left(-\frac{1}{2}\right) \right]_{-1}^{1} = 0 - 0 = 0$$

DISPARI INTEGRATA IN INTERVALLO SIMMETRICO $\Rightarrow$ 0.

$$x_B = \frac{\int_\Omega x\, dxdy}{\int_\Omega dxdy} = 0 \qquad \int_\Omega y\, dxdy = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} y\, dy = \int_{-1}^{1} \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1}{2} \cdot (1-x^2)\, dx = \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$y_B = \frac{\int_\Omega y\, dxdy}{\int_\Omega dxdy} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}.$$

Ω è convesso e infatti $\left(0, \frac{4}{3\pi}\right) \in \Omega$

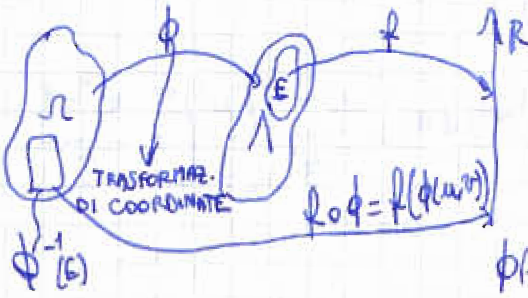
$x=0$ è asse di simmetria per Ω e infatti $\left(0, \frac{4}{3\pi}\right) \in$ asse.

TEOREMA DEL CAMBIO DI VARIABILI

Sia $\Omega, \Lambda \subseteq \mathbb{R}^2$ aperti, $\phi: \Omega \to \Lambda$ di classe C^1 biettiva, $0 < |\det J_\phi(u,v)| < +\infty$ $\forall u,v \in \Omega$

Sia $f: E \subset \Lambda \to \mathbb{R}$ continua ed E misurabile. Allora:

- f integrabile su E $\left(\exists \int_E f(x,y)\, dxdy\right)$.
- $f(\phi(u,v)) \cdot |\det J_\phi(u,v)|$ è integrabile su $\phi^{-1}(E)$ (controimmagine)
- $\int_E f(x,y)\, dxdy = \int_{\phi^{-1}(E)} f(\phi(u,v)) \cdot |\det J_\phi(u,v)|\, du\, dv$.



$$J_\phi(u,v) = \begin{pmatrix} \nabla\phi_1(u,v) \\ \nabla\phi_2(u,v) \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial v} \\ \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v} \end{pmatrix}$$

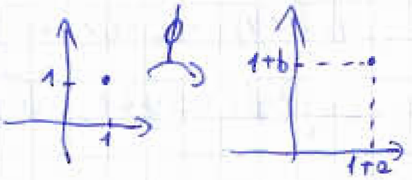
$\phi(u,v) = (\phi_1(u,v), \phi_2(u,v))$

$\phi : \mathbb{R}^2 \to \mathbb{R}^2$ è una traslazione ed $E \subseteq \mathbb{R}^2$ è misurabile, allora $m(\phi(E)) = m(E)$

DIM.

$\phi : \mathbb{R}^2 \to \mathbb{R}^2$

$(u,v) \to (u+a, v+b)$ $(a,b) \in \mathbb{R}^2$ fissati



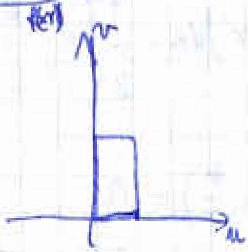
$$J_\phi(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad |\det J_\phi(u,v)| = 1.$$

$$m(\phi(E)) = \int_{\phi(E)} 1 \, dx\, dy \overset{(TCV)}{=} \int_{\underbrace{\phi^{-1}(\phi(E))}_{E}} \overbrace{|\det J_\phi(u,v)|}^{1} du\, dv = \int_E du\, dv = m(E) \quad \blacksquare$$

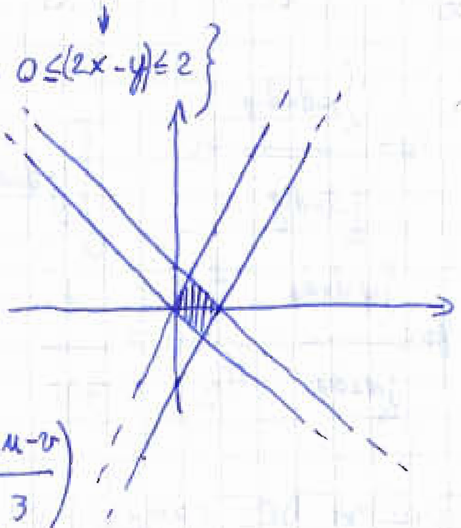
$f(x,y) = 1$

$$\int_E \underbrace{(x+y)\, e^{2x-y}}_{f(x,y)} dx\, dy \qquad E = \{(x,y) \in \mathbb{R}^2 : 0 \le \overset{u}{(x+y)} \le 1 ,\ 0 \le \overset{v}{(2x-y)} \le 2\}$$

$$\begin{cases} y \ge -x \\ y \le 1-x \\ y \le 2x \\ y \ge 2x-2 \end{cases}$$



$\xleftarrow{\phi}$



$$\begin{cases} x+y = u \\ 2x - y = v \end{cases} \qquad 3x = u+v \qquad x = \frac{u+v}{3} \qquad \begin{cases} x = \frac{u+v}{3} \\ y = \frac{2u-v}{3} \end{cases} \qquad \phi(u,v) = \Big(\underset{\phi_1}{\frac{u+v}{3}} , \underset{\phi_2}{\frac{2u-v}{3}} \Big)$$

$$\phi^{-1}(E) = \{(u,v) : 0 \le u \le 1 ,\ 0 \le v \le 2\} \qquad J_\phi = \begin{pmatrix} \nabla \phi_1 \\ \nabla \phi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} , & \frac{1}{3} \\ \frac{2}{3} , & -\frac{1}{3} \end{pmatrix}$$

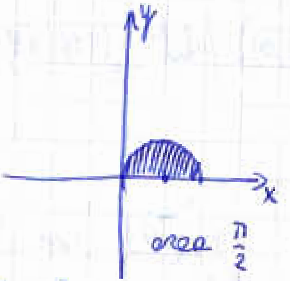
$$|\det J_\phi(u,v)| = \left| -\frac{1}{9} - \frac{2}{9} \right| = \left| -\frac{1}{3} \right| = \frac{1}{3}$$

$$\int_E f(x,y)\, dx\, dy = \int_{\phi^{-1}(E)} f(\phi(u,v)) \cdot |\det J(\phi(u,v))|\, du\, dv = \int_{\phi^{-1}(E)} u \cdot e^v \cdot \frac{1}{3}\, du\, dv =$$

$$= \int_0^1 du \int_0^2 u \cdot e^v \cdot \frac{1}{3}\, dv = \int_0^1 \frac{u}{3}\, du \left[e^v \right]_0^2 = \frac{1}{3} \int_0^1 u \cdot (e^2 - 1)\, du = \frac{e^2-1}{3} \left[\frac{u^2}{2} \right]_0^1 = \frac{e^2-1}{6}$$

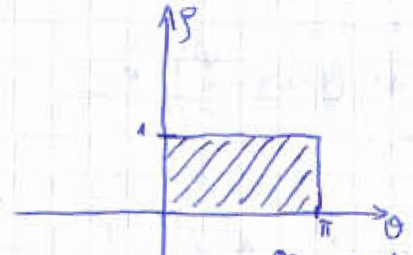
ESERCIZIO

$$\int_C x^2 y\,dx\,dy \qquad C=\{(x,y)\in\mathbb{R}^2 : (x-1)^2+y^2\le 1,\ y\ge 0\}$$



$$\underset{\overset{\uparrow}{x_c}}{\Theta}(\rho,\theta)=(\overset{}{\underset{\uparrow\ x_c}{1+\rho\cos\theta}},\ \underset{\uparrow\ y_c}{\rho\,\text{sen}\,\theta}) \qquad \begin{cases}(1+\rho\cos\theta-1)^2+\rho^2\text{sen}^2\theta\le 1\\ \rho\,\text{sen}\,\theta\ge 0\end{cases} \qquad \rho\ge 0\quad \theta\in[0,2\pi]$$

$$\begin{cases}\cancel{\rho^2\cos^2\theta}+\rho^2-\cancel{\rho^2\cos^2\theta}\le 1\\ \rho\,\text{sen}\,\theta\ge 0 \quad \text{sen}\,\theta\ge 0 \text{ perchè } \rho\ge 0\end{cases} \qquad \begin{cases}\rho^2\le 1\\ \text{sen}\,\theta\ge 0\end{cases} \qquad \Theta^{-1}(C)=\{0\le\rho\le 1,\ 0\le\theta\le\pi\}$$



$$J_\Theta(\rho,\theta)=\begin{vmatrix}\frac{\partial\Theta_1}{\partial\rho} & \frac{\partial\Theta_1}{\partial\theta}\\ \frac{\partial\Theta_2}{\partial\rho} & \frac{\partial\Theta_2}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -\rho\,\text{sen}\,\theta\\ \text{sen}\,\theta & \rho\cos\theta\end{vmatrix}$$

area π ⇒ tengo conto di J.

$$|\det J_\Theta(\rho,\theta)|=|\rho\cos^2\theta+\rho\,\text{sen}^2\theta|=|\rho|\ge 0=\rho.$$

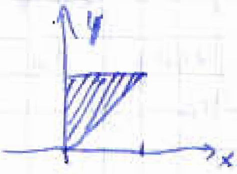
$$\int_{\Theta^{-1}(C)} f(\Theta(\rho,\theta))\cdot|\det J_\Theta(\rho,\theta)|\,d\theta\,d\rho=\int_{\Theta^{-1}(C)}(1+\rho\cos\theta)^2\cdot\rho\,\text{sen}\,\theta\cdot\rho\,d\theta\,d\rho=$$

$$=\int_0^\pi d\theta\int_0^1(1+\rho^2\cos^2\theta+2\rho\cos\theta)\cdot\rho^2\text{sen}\,\theta\,d\rho=\int_0^\pi\text{sen}\,\theta\,d\theta\cdot\left(\int_0^1\rho^2 d\rho+\int_0^1\rho^4\cos^2\theta\,d\rho+\int_0^1 2\rho^3\cos\theta\,d\rho\right)=$$

$$=\int_0^\pi\text{sen}\,\theta\,d\theta\left(\frac{1}{3}+\cos^2\theta\cdot\frac{1}{5}+\cancel{2}\cos\theta\,\frac{1}{\cancel{4}2}\right)=\int_0^\pi\frac{\text{sen}\,\theta}{3}d\theta+-\int_0^\pi\frac{\text{sen}\,\theta\cos^2\theta\,d\theta}{5}+\int_0^\pi\frac{\text{sen}\,\theta\cos\theta}{2}d\theta=$$

$$=\frac{1}{3}\Big[-\cos\theta\Big]_0^\pi-\frac{1}{5}\left[\frac{\cos^3\theta}{3}\right]_0^\pi+\frac{1}{2}\left[\frac{\text{sen}^2\theta}{2}\right]_0^\pi=\frac{1}{3}(1+1)-\frac{1}{5}\left(-\frac{1}{3}-\frac{1}{3}\right)+\cancel{\frac{1}{2}(0)}=\frac{2}{3}+\frac{2}{15}=\frac{4}{5}.$$

$$\int_T xe^x\,dx\,dy \qquad T=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 2,\ x\le y\le 2$$



$$\Theta(\rho,\theta)=(\rho\cos\theta,\rho\,\text{sen}\,\theta) \qquad \underbrace{0\le\overset{\ge 0}{\rho}\cos\theta\le 2}_{\cos\theta\ge 0} \qquad \underbrace{\rho\cos\theta\le\rho\,\text{sen}\,\theta\le 2}$$

$\rho\cos\theta\ge 0$ ma $\rho\,\text{sen}\,\theta\ge\rho\cos\theta \Rightarrow \rho\,\text{sen}\,\theta\ge 0 \Rightarrow \underline{\text{sen}\,\theta\ge 0}$

$\rho\cos\theta\le\rho\,\text{sen}\,\theta \quad \underline{tg\,\theta\ge 1}$ $\Longrightarrow \theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

$\det J=\rho.$

$\rho\,\text{sen}\,\theta\le 2 \quad \rho\le\frac{2}{\text{sen}\,\theta}$

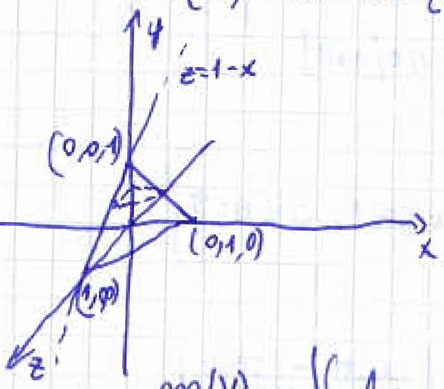
$$T=\left\{(\rho,\theta):\frac{\pi}{4}\le\theta\le\frac{\pi}{2},\ 0\le\rho\le\frac{2}{\text{sen}\,\theta}\right\}$$

$$\int_{\Phi^{-1}(T)} f(\Phi(\rho,\theta)) \cdot |\det J_{\Phi}(\rho,\theta)| \, d\rho \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{3\,\mathrm{sen}\theta} \rho\cos\theta \cdot e^{\rho\cos\theta} \cdot \rho \, d\rho$$

NON FARE perchè si ottiene una roba più difficile di prima.

26/05/2008

Calcolare $m(V)$ con $V = \{(x,y,z) : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$



$z = 1-x-y \quad \Omega = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$

$V : \{(x,y,z) : (x,y) \in \Omega,\ 0 \leq z \leq f(x,y)\}$

misura / volume in questo caso

$$m(V) = \iint_{\Omega} f(x,y)\,dx\,dy = \int_0^1 dx \int_0^{1-x} (1-x-y)\,dy = \int_0^1 dx \cdot \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} =$$

$$= \int_0^1 \left[1-x-x+x^2 - \frac{1}{2}(1+x^2-2x)\right] dx = \int_0^1 \left[\frac{1}{2} - x + \frac{1}{2}x^2\right] dx = \left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{6}$$ oppure.

Fisso z costante per affettare la figura con piani orizzontali:

$0 \leq z \leq 1 \quad (x,y) \in T_z \quad T_z = \{(x,y) : 0 \leq x \leq 1-z,\ 0 \leq y \leq 1-x-z\}$

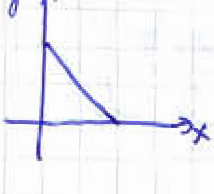
$V = \{(x,y,z) : 0 \leq z \leq 1,\ 0 \leq x \leq 1-z,\ 0 \leq y \leq 1-x-z\}$

$$m(V) = \int_V dx\,dy\,dz = \int_0^1 dz \left[\iint_{T_z} dx\,dy\right] = \int_0^1 dz \int_0^{1-z} dx \int_0^{1-x-z} dy = \int_0^1 dz \int_0^{1-z} (1-x-z)\,dx = \int_0^1 \left[x - \frac{x^2}{2} - zx\right]_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z) - \frac{(1-z)^2}{2} - z(1-z)\right] dz = \frac{1}{6}.$$ Oppure

$\prod_{xy}(V) = T_0 = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\}$

PROIEZIONE DI V SU xy PIANO



$$\int_{V} dx\,dy\,dz = \int_{T_0} dx\,dy \int_0^{1-x-y} dz = \int_{T_0} (1-x-y)\,dx\,dy = \frac{1}{6}$$ da prima

---

$E \subseteq \mathbb{R}^3 \quad E = \{(x,y,z) : a \leq x \leq b,\ (y,z) \in E_x\}$ DOMINIO SEMPLICE RISPETTO ALL'ASSE X

TEOREMA INTEGRAZIONE PER STRATI

Data $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ continua e limitata e sia $E \subseteq \Omega$ dominio semplice rispetto all'asse x; allora:

$U(x) = \int_{E_x} f(x,y,z)\, dy\, dz$ è continua e $\int_E f\, dx\, dy\, dz = \int_a^b U(x)\, dx$

DEF.

$E \subset \mathbb{R}^3$ misurabile e limitato si dice normale rispetto al piano XY se

- $\Pi_{xy}(E)^{=V}$ è misurabile
- $\exists\, a, b : V \to \mathbb{R}$ $a(x,y) \leq b(x,y)$ $(x,y \in V)$ tali che

$E = \{(x,y,z) : (x,y) \in V = \Pi_{xy}(E),\ a(x,y) \leq z \leq b(x,y)\}$

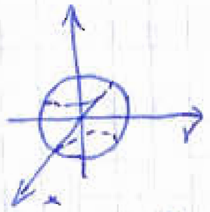
TEOREMA INTEGRAZIONE PER FILI

$f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ continua e limitata, $E \subseteq \Omega$ dominio normale rispetto al piano xy

allora $U(x,y) = \int_{a(x,y)}^{b(x,y)} f(x,y,z)\, dz$ è continua e $\int_E f\, dx\, dy\, dz = \int_{\Pi_{xy}(E)} U(x,y)\, dx\, dy$.

ESERCIZIO

Calcolare il volume della sfera $B = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

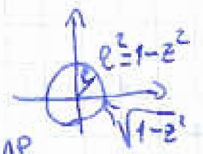


B è normale rispetto al piano (XY)

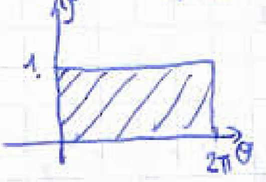
$\Pi_{xy}(B) = \{(x,y) : x^2+y^2 \leq 1\}$ $\quad B = \{(x,y,z) : (x,y) \in \Pi_{xy}(B),\ -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}\}$

$m(B) = \int_{\Pi_{xy}(B)} dx\, dy \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz$ integrazione per fili...

$m(B) = \int_{-1}^{1} dz \int_{B_z} dx\, dy$ $\quad B_z = \{(x,y) : x^2+y^2 \leq 1-z^2\}$



integrazione per strati



$\int_{B_z} dx\, dy \overset{?}{=}$ $\quad \Theta : (\rho, \theta) \to (\rho\cos\theta, \rho\,\mathrm{sen}\,\theta)$

$\Theta^{-1}(B_z)$ $\{(\rho,\theta) : \rho\cos^2\theta + \rho\,\mathrm{sen}^2\theta \leq 1-z^2\}$

$= \{(\rho,\theta) : 0 \leq \rho^2 \leq 1-z^2\} = \{(\rho,\theta) : 0 \leq \rho \leq \sqrt{1-z^2}\}$ e $\theta \in [0, 2\pi]$ (non ho condizioni)

$|\det J_{\Theta}(\rho,\theta)| = \rho$ $\quad \int_{B_z} dx\, dy = \int_{\Theta^{-1}(B_z)} \rho\, d\theta\, d\rho = \int_0^{2\pi} d\theta \int_0^{\sqrt{1-z^2}} \rho\, d\rho = \int_0^{2\pi} d\theta\, \frac{1-z^2}{2} = \pi(1-z^2)$

$$m(B)=\int_{-1}^{1} dz\left(\pi(1-z^2)\right)=\pi\int_{-1}^{1}(1-z^2)dz=\pi\left[z-\frac{z^3}{3}\right]_{-1}^{1}=\pi\left(1-\frac{1}{3}+1+\frac{1}{3}\right)=\frac{4}{3}\pi$$

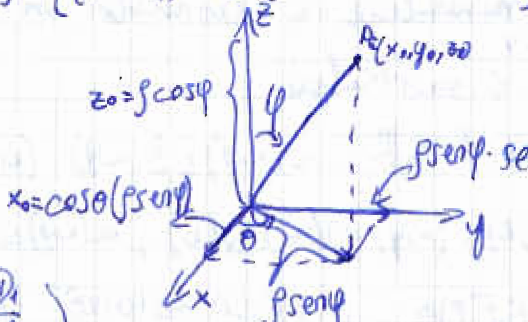
Trasformo la sfera in un parallelepipedo con le coordinate sferiche.

$$\textcircled{H}:(\rho,\theta,\varphi)\longrightarrow(\underbrace{\rho\cos\theta\,\text{sen}\,\varphi}_{x},\underbrace{\rho\,\text{sen}\,\theta\,\text{sen}\,\varphi}_{y},\underbrace{\rho\cos\varphi}_{z})$$

$$\textcircled{H}^{-1}(B)=\{(\rho,\theta,\varphi):\rho^2\cos^2\theta\,\text{sen}^2\varphi+\rho^2\text{sen}^2\theta\,\text{sen}^2\varphi+\rho^2\cos^2\varphi\le 1\}=$$

$$=\{(\rho,\theta,\varphi):(\rho^2\cos^2\theta+\rho^2\text{sen}^2\theta)\text{sen}^2\varphi+\rho^2\cos^2\varphi\le 1\}=\{(\rho,\theta,\varphi):\rho^2(\text{sen}^2\theta+\cos^2\theta)\le 1\}=$$

$$=\{(\rho,\theta,\varphi):0<\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\pi\}$$



φ: angolo tra $\vec{OP}$ e $z>0$.

$0\le\varphi\le\pi$ dopo trovo angoli che conosco già

$0\le\theta\le 2\pi$ come prima

$0\le\rho$

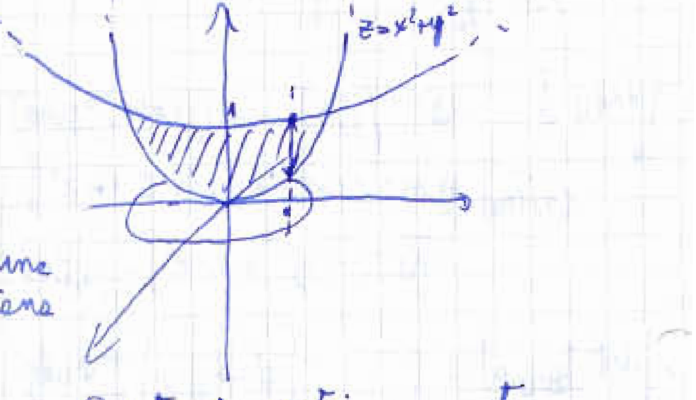
$$J_{\textcircled{H}}=\begin{pmatrix}\frac{\partial\textcircled{H}_1}{\partial\rho} & \frac{\partial\textcircled{H}_1}{\partial\theta} & \frac{\partial\textcircled{H}_1}{\partial\varphi}\\ \dots & & \dots\\ \dots & \dots & \dots\end{pmatrix}=$$

$$=\begin{pmatrix}\cos\theta\,\text{sen}\,\varphi & -\rho\,\text{sen}\,\theta\,\text{sen}\,\varphi & \rho\cos\theta\cos\varphi\\ \text{sen}\,\theta\,\text{sen}\,\varphi & \rho\cos\theta\,\text{sen}\,\varphi & \rho\,\text{sen}\,\theta\cos\varphi\\ \cos\varphi & 0 & -\rho\,\text{sen}\,\varphi\end{pmatrix}\quad \det J_{\textcircled{H}}(\rho,\theta,\varphi)=\rho^2\underbrace{\text{sen}\,\varphi}_{>0\ \forall\varphi\in[0,\pi]}$$

$$\int_B dx\,dy\,dz=\int_{\textcircled{H}^{-1}(B)}\left|\det J_{\textcircled{H}}(\rho,\theta,\varphi)\right|d\rho\,d\theta\,d\varphi=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1\rho^2\text{sen}\,\varphi\,d\rho=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\left[\frac{\rho^3}{3}\text{sen}\,\varphi\right]_{\rho=0}^{\rho=1}=$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi}\frac{\text{sen}\,\varphi}{3}d\varphi=\frac{1}{3}\int_0^{2\pi}d\theta\left[-\cos\varphi\right]_0^{\pi}=\frac{1}{3}\int_0^{2\pi}2\,d\theta=\frac{2}{3}\cdot 2\pi=\frac{4}{3}\pi$$

Calcolare la misura (volume) di $V\subseteq\mathbb{R}^3$ delimitata da $S_1: z=x^2+y^2$ e $S_2: z=1+\frac{3}{4}x^2+\frac{8}{9}y^2$



$$\begin{cases}z=x^2+y^2\\ z=1+\frac{3}{4}x^2+\frac{8}{9}y^2\end{cases}\qquad x^2+y^2=1+\frac{3}{4}x^2+\frac{8}{9}y^2$$

cilindro $\rightarrow$ $\begin{cases}A & \frac{1}{4}x^2+\frac{1}{9}y^2=1\\ B & z=x^2+y^2\end{cases}$ non è una curva piana $A\cap B$

$(0,3,9)$ $(0,-3,9)$ $(2,0,4)$ $(-2,0,4)$ Questi 4 punti non stanno su nessun piano.

se $x=0$, $y=\pm3$, $z=9$; se $y=0$, $x=\pm2$, $z=4$

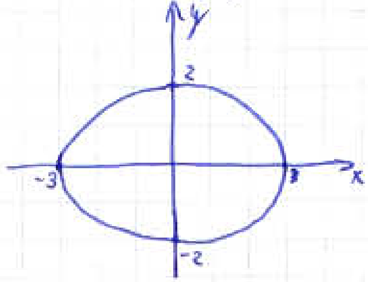
Da questo ottengo la proiezione : $\Pi_{xr}(V)=\left\{(x,y) : \frac{x^2}{4}+\frac{y^2}{9}\leq 1\right\}$

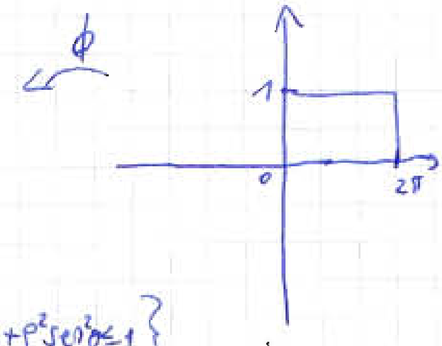
$V=\left\{(x,y,z) : (x,y)\in\Pi_{xr}(V),\ x^2+y^2\leq z\leq 1+\frac{3}{4}x^2+\frac{8}{9}y^2\right\}$

$$m(V)=\int_V dx\,dy\,dz=\int_{\Pi_{xr}(V)} dx\,dy\left(\int_{x^2+y^2}^{1+\frac{3}{4}x^2+\frac{8}{9}y^2} dz\right)=\int_{\Pi_{xy}(V)}\left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dx\,dy$$

$$\Pi_{xr}(V)=\left\{(x,y) : \frac{x^2}{4}+\frac{y^2}{9}\leq 1\right\}$$

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2\leq 1$$





$$\begin{cases}\frac{x}{2}=\varrho\cos\theta\\ \frac{y}{3}=\varrho\,\mathrm{sen}\,\theta\end{cases}\qquad\begin{cases}x=2\varrho\cos\theta\\ y=3\varrho\,\mathrm{sen}\,\theta\end{cases}$$

$$\phi^{-1}[\Pi_{xr}(V)]=\left\{(\varrho,\theta) : \varrho^2\cos^2\theta+\varrho^2\mathrm{sen}^2\theta\leq 1\right\}$$
$$=\left\{(\varrho,\theta) : 0\leq\varrho\leq 1,\ \theta\in[0,2\pi]\right\}$$

$$J_\phi=\begin{pmatrix}2\cos\theta & -2\varrho\,\mathrm{sen}\,\theta\\ 3\,\mathrm{sen}\,\theta & 3\varrho\cos\theta\end{pmatrix}\qquad \det J_\phi(\varrho,\theta)=6\varrho\cos^2\theta+6\varrho\,\mathrm{sen}^2\theta=6\varrho$$

$$\int_{\Pi_{xr}(V)}1-\left(\frac{x}{2}\right)^2-\left(\frac{y}{3}\right)^2dx\,dy=\int_{\phi^{-1}[\Pi_{xr}(V)]}(1-\varrho^2)\cdot 6\varrho\,d\varrho\,d\theta=\int_0^{2\pi}d\theta\int_0^1 6\varrho-6\varrho^3d\varrho=\int_0^{2\pi}\left[3\varrho^2-\frac{3}{2}\varrho^4\right]_0^1d\theta=\int_0^{2\pi}\frac{3}{2}d\theta=3\pi$$

Calcolare $m(E)$ $\quad E=\left\{(x,y,z) : x^2+y^2\leq z\leq 8-x^2-3y^2\right\}$

$$\begin{cases}z=x^2+y^2\\ z=8-x^2-3y^2\end{cases}\qquad x^2+y^2=8-x^2-3y^2\qquad\begin{cases}2x^2+4y^2=8\\ z=x^2+y^2\end{cases}\qquad\begin{cases}x^2+2y^2=4\\ z=x^2+y^2\end{cases}$$

CILINDRO
⇓
proiezione

$\Pi_{xr}(E)=\left\{(x,y) : \frac{x^2}{4}+\frac{y^2}{2}\leq 1\right\}$ $\quad E=\left\{(x,y,z) : (x,y)\in\Pi_{xr}(E),\ x^2+y^2\leq z\leq 8-x^2-3y^2\right\}$

$$\int_E dx\,dy\,dz=\int_{\Pi_{xr}(E)}dx\,dy\int_{x^2+y^2}^{8-x^2-3y^2}dz=\int_{\Pi_{xr}(E)}(8-2x^2-4y^2)\,dx\,dy$$

$$\begin{cases}\frac{x}{2}=\varrho\cos\theta\\ \frac{y}{\sqrt{2}}=\varrho\,\mathrm{sen}\,\theta\end{cases}\qquad \det J_\phi=2\sqrt{2}\varrho\ \ldots$$

ESERCITAZIONI

12/03/2008

1.8.6.

1 giro, verso antiorario $\{(x,y)\in\mathbb{R}^2 : x^2+9y^2=1\}$ $P_0=(-1,0)$

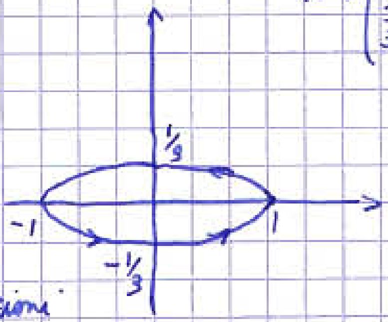
SOSTEGNO

$$x^2+\left(\frac{y}{\frac{1}{3}}\right)^2=1$$

Trovare una parametrizzazione.

CURVA → equazione

SOSTEGNO → grafico è un'ellisse



A un sostegno corrispondono ∞ parametrizzazioni

$$\begin{cases} x=a\cos t \\ y=b\,\mathrm{sen}\,t \end{cases} \quad t\in[0,2\pi]$$ equazione parametrica ellisse

PUNTO DI PARTENZA $t=0$ $\begin{cases} x=a \\ y=0 \end{cases}$

$t=\frac{\pi}{2}$ $\begin{cases} x=0 \\ y=b \end{cases}$ verso antiorario

SEGNI DI a E b CONCORDI → VERSO ANTIORARIO

SEGNI DI a E b DISCORDI → VERSO ORARIO

$$\begin{cases} x=\overset{a}{\downarrow}{1}\cdot\cos t \\ y=\underset{\overset{\uparrow}{b}}{\frac{1}{3}}\,\mathrm{sen}\,t \end{cases}$$

ora sistemo il punto di partenza $P_0(-1,0)$ $\begin{cases} -1=\cos t \\ 0=\frac{1}{3}\,\mathrm{sen}\,t \end{cases}$ $\begin{cases} \mathrm{sen}\,t=0 \\ \cos t=-1 \end{cases}$ $t=\pi \Rightarrow t\in[\pi,3\pi]$ per un giro completo.

Scrivere eq. parametrica e cartesiana della tangente alla curva in $P\left(\frac{\sqrt{3}}{2},\frac{1}{6}\right)$

$$\begin{cases} \frac{\sqrt{3}}{2}=\cos t \\ \frac{1}{6}=\frac{1}{3}\,\mathrm{sen}\,t \end{cases} \quad \begin{cases} \mathrm{sen}\,t=\frac{1}{2} \\ \cos t=\frac{\sqrt{3}}{2} \end{cases} \quad t=\frac{\pi}{6} \quad \varphi'\left(\frac{\pi}{6}\right)=\left(x'\left(\tfrac{\pi}{6}\right),y'\left(\tfrac{\pi}{6}\right)\right)$$

$x'=-\mathrm{sen}\,t$

$y'=\frac{1}{3}\cos t$

$$\varphi'\left(\frac{\pi}{6}\right)=\left(-\frac{1}{2},\frac{\sqrt{3}}{6}\right)\xrightarrow[\text{della retta}]{\text{equazione}}$$

PARAMETRICA

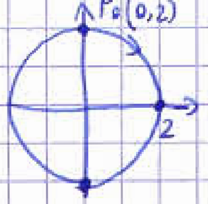
$$\begin{cases} x=x_0+x'(t)\cdot t \\ y=y_0+y'(t)\cdot t \end{cases} \quad \begin{cases} x=\frac{\sqrt{3}}{2}-\frac{1}{2}t \\ y=\frac{1}{6}+\frac{\sqrt{3}}{6}t \end{cases}$$

CARTESIANA

$$\frac{x-x_0}{x'(t)}=\frac{y-y_0}{y'(t)} \qquad \frac{x-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=\frac{y-\frac{1}{6}}{\frac{\sqrt{3}}{6}} \qquad -2x+\sqrt{3}=\frac{6y}{\sqrt{3}}-\frac{1}{\sqrt{3}} \qquad -2\sqrt{3}x+3=6y-1$$

$$6y+2\sqrt{3}x-4=0 \qquad 3y+\sqrt{3}x-2=0$$

Circonferenza



1 giro verso orario

$x^2+y^2=R^2 \quad x^2+y^2=4 \quad a=2$

$$\begin{cases} x = -a\cos t \\ y = a\,\text{sen}\,t \end{cases} \quad t\in[0,2\pi]$$

$t=0 \quad P_0(-2,0)$

$t=\frac{\pi}{2} \quad P_0(0,2)$

$$\begin{cases} 0 = -2\cos t \\ 2 = 2\,\text{sen}\,t \end{cases} \quad \begin{cases} \cos t = 0 \\ \text{sen}\,t = 1 \end{cases} \quad t=\frac{\pi}{2} \Rightarrow t\in\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$$

$$\begin{cases} x = -2\cos t \\ y = 2\,\text{sen}\,t \end{cases} \quad t\in\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$$

PARAMETRIZZAZIONI DA RICORDARE

$$\begin{cases} x = R\cos t \\ y = R\,\text{sen}\,t \end{cases} \quad t\in[0,2\pi]$$

CIRCONFERENZA CENTRATA NELL'ORIGINE

$$\begin{cases} x = x_c + R\cos t \\ y = y_c + R\,\text{sen}\,t \end{cases} \quad t\in[0,2\pi]$$

CIRCONFERENZA CENTRATA IN $C(x_c,y_c)$

$$\begin{cases} x = a\cos t \\ y = b\,\text{sen}\,t \end{cases} \quad t\in[0,2\pi]$$

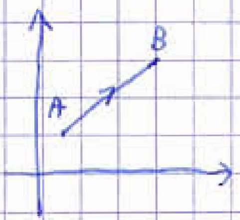
ELLISSE CENTRATA NELL'ORIGINE

$$\begin{cases} x = x_C + a\cos t \\ y = y_c + b\,\text{sen}\,t \end{cases} \quad t\in[0,2\pi]$$

ELLISSE CENTRATA IN $C(x_c,y_c)$

$$\begin{cases} x = t \\ y = mt+q \end{cases} \quad t\in\mathbb{R}$$

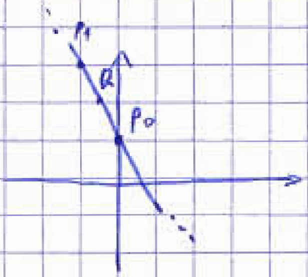
RETTA $y=mx+q$



$$\begin{cases} x = x_A + t(x_B-x_A) \\ y = y_A + t(y_B-y_A) \end{cases} \quad t\in[0,1]$$

SEGMENTO DA A A B.

1.8.7



5) retta passante per $P_1(0,1)$ $P_2(-1,3)$ $Q\left(-\frac{1}{2},2\right)$

Scegliamo un verso di percorrenza ↖

$$\begin{cases} x = 0 + t(-1-0) \\ y = 1 + t(3-1) \end{cases} \quad \begin{cases} x = -t \\ y = 2t+1 \end{cases} \quad t\in\mathbb{R}$$

Trovare il vettore tangente, il versore tangente e normale in Q.

$$\begin{cases} x' = -1 \\ y' = 2 \end{cases} \quad \varphi'(t) = (-1,2) \quad |\varphi'(t)| = \sqrt{1+4} = \sqrt{5} \quad \underset{\text{versore tangente}}{\vec{T}} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

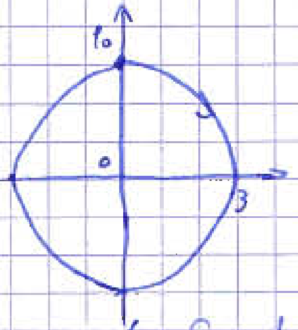
$$\vec{N} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = (y_T, -x_T)$$

versore con verso esterno

$C(0,0)$ verso orario

$R=3 \qquad P_0(0,3)$

$Q\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$



$$\begin{cases} x=-3\cos t \\ y=3\operatorname{sen} t \end{cases} \quad t\in\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$$

$$\begin{cases} x'=3\operatorname{sen} t \\ y'=3\cos t \end{cases} \qquad \begin{cases} \frac{3\sqrt{2}}{2}=-3\cos t \\ -\frac{3\sqrt{2}}{2}=3\operatorname{sen} t \end{cases} = \begin{cases} \operatorname{sen} t=\frac{-\sqrt{2}}{2} \\ \cos t=-\frac{\sqrt{2}}{2} \end{cases} \quad t=\frac{5}{4}\pi$$

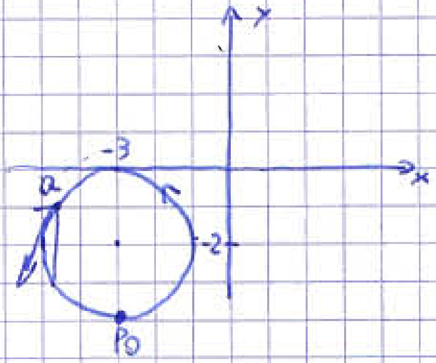
$$\varphi'\left(\frac{5}{4}\pi\right)=\left(-\frac{3\sqrt{2}}{2}; -\frac{3\sqrt{2}}{2}\right) \qquad |\varphi'(t)|=\sqrt{\frac{18}{4}+\frac{18}{4}}=3 \qquad \vec{T}=\left(\frac{-\frac{3\sqrt{2}}{2}}{3}, \frac{-\frac{3\sqrt{2}}{2}}{3}\right)=\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\vec{N}=\left(-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}\right)$$

$C=(-3,-2) \qquad P_0=(-3,-4)$

$R=2$

verso antiorario



$$\begin{cases} x=-3+2\cos t \\ y=-2+2\operatorname{sen} t \end{cases} \quad t\in\left[-\frac{\pi}{2}, \frac{3}{2}\pi\right]$$

$$\begin{cases} -3=-3+2\cos t \\ -4=-2+2\operatorname{sen} t \end{cases} \quad \begin{cases} \cos t=0 \\ \operatorname{sen} t=-1 \end{cases} \quad t=\frac{3}{2}\pi \qquad Q=(-3-\sqrt{3}, -1)$$

$$\begin{cases} x'=-2\operatorname{sen} t \\ y'=2\cos t \end{cases} \qquad \begin{cases} -3-\sqrt{3}=-3+2\cos t \\ -1=-2+2\operatorname{sen} t \end{cases} \quad \begin{cases} \cos t=-\frac{\sqrt{3}}{2} \\ \operatorname{sen} t=\frac{1}{2} \end{cases} \quad t=\frac{5}{6}\pi$$

$$\begin{cases} x'=-2\operatorname{sen}\frac{5}{6}\pi \\ y'=2\cos\frac{5}{6}\pi \end{cases} \quad \varphi'\left(\frac{5}{6}\pi\right)=\left(-1, -\sqrt{3}\right) \qquad \left|\varphi'\left(\frac{5}{6}\pi\right)\right|=\sqrt{1+3}=2 \quad \vec{T}=\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad \vec{N}=\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Ellisse $C(1,1)$ $a=2$ $b=1$ verso antiorario $P_0(3,1)$ $Q\left(1+\sqrt{3}, \frac{3}{2}\right)$

$$\begin{cases} x=1+2\cos t \\ y=1+\operatorname{sen} t \end{cases} \quad t\in[0,2\pi] \qquad \begin{cases} 3=1+2\cos t \\ 1=1+\operatorname{sen} t \end{cases} \quad \begin{cases} \cos t=1 \\ \operatorname{sen} t=0 \end{cases} \quad t=0$$

$$\begin{cases} x'=-2\operatorname{sen} t \\ y'=\cos t \end{cases} \qquad \begin{cases} 1+\sqrt{3}=1+2\cos t \\ \frac{3}{2}=1+\operatorname{sen} t \end{cases} \quad \begin{cases} \cos t=\frac{\sqrt{3}}{2} \\ \operatorname{sen} t=\frac{1}{2} \end{cases} \quad t=\frac{\pi}{6} \qquad \varphi'\left(\frac{\pi}{6}\right)=\left(-1, \frac{\sqrt{3}}{2}\right)$$

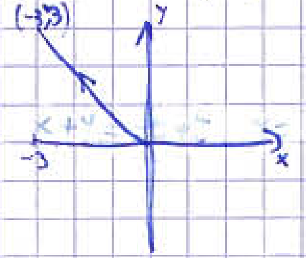
L'IMPORTANTE E` TROVARE sent E cost, NON L'ANGOLO.

$\left|\varphi'\left(\frac{\pi}{6}\right)\right| = \sqrt{1+\frac{3}{4}} = \frac{\sqrt{7}}{2}$ $\quad \vec{T} = \left(-\frac{\frac{1}{\sqrt{7}}}{2} ; \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{7}}{2}}\right) = \left(-\frac{2}{\sqrt{7}} ; \frac{\sqrt{3}}{\sqrt{7}}\right)$ $\quad \vec{N} = \left(\frac{\sqrt{3}}{\sqrt{7}} ; \frac{2}{\sqrt{7}}\right)$

1.8.11

Disegnare il sostegno della curva, con verso, e la lunghezza

$\varphi : [0,5] \to \mathbb{R}^2$



① $\begin{cases} x = -t \\ y = t \end{cases} \quad t \in [0,3]$

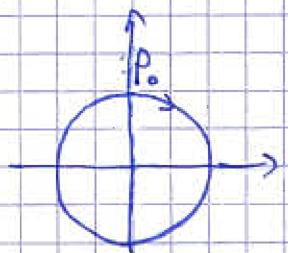
② $\begin{cases} x = 3\cos(\pi t) \\ y = 3 - 3\operatorname{sen}(\pi t) \end{cases} \quad t \in (3,4]$ $\quad C = (0,3) \quad R = 3$ verso orario

$P_0 = (-3,3) \quad P_1 = (3,3)$

③ $\begin{cases} x = 3 - 3\operatorname{sen}\left(\frac{\pi}{2}t\right) \\ y = 3\cos\left(\frac{\pi}{2}t\right) \end{cases} \quad t \in (4,5]$ $\quad C = (3,0) \quad R = 3$ verso antiorario

$P_0(3,3) \quad P_1(0,0)$

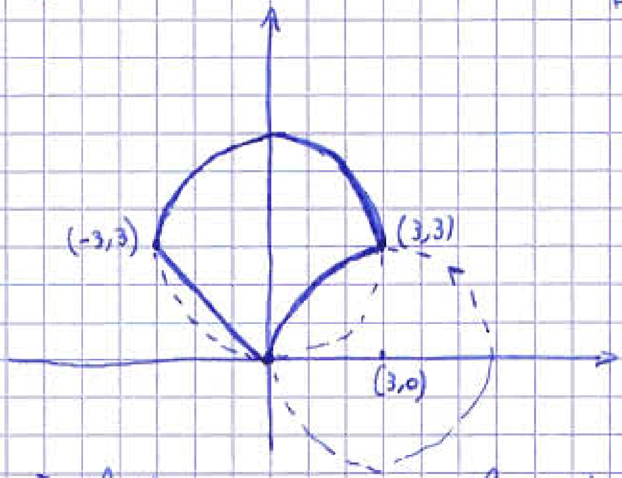
$\begin{cases} x = R\operatorname{sen}t \\ y = R\cos t \end{cases} \quad t \in [0, 2\pi]$



SEGNI CONCORDI → VERSO ORARIO

SEGNI DISCORDI → VERSO ANTIORARIO

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$



In questo caso si può calcolare per via elementare

L_1 = lunghezza diagonale quadrato di lato $3 = 3\sqrt{2}$

L_2 = lunghezza di una semicfr di raggio $3 = \frac{1}{2} \cdot 2\pi \cdot 3 = 3\pi$

L_3 = " " un quarto di cfr di raggio $3 = \frac{1}{4} \cdot 2\pi \cdot 3 = \frac{3}{2}\pi$

$L = 3\sqrt{2} + 3\pi + \frac{3}{2}\pi = 3\sqrt{2} + \frac{9}{2}\pi$

1.8.16

Trovare una curva avente sostegno $A = \{(x,y) \in \mathbb{R}^2 : x^2 = y^3, -1 \leq x \leq 1\}$

$y = \sqrt[3]{x^2} \quad \begin{cases} x = t \\ y = \sqrt[3]{t^2} \end{cases} \quad t \in [-1,1]$ $\quad$ Trovare la lunghezza

$\varphi' = \begin{cases} x' = 1 \\ y' = \frac{2}{3} t^{\frac{2}{3}-1} \end{cases} = \begin{cases} x' = 1 \\ y' = \frac{2}{3} t^{-\frac{1}{3}} \end{cases}$ $\quad L = \int_{-1}^{1} \sqrt{1 + \frac{4}{9} t^{-\frac{2}{3}}}\, dt = \int_{-1}^{1} \sqrt{\frac{9t^{2/3} + 4}{9t^{2/3}}}\, dt =$

$$= \int_{-1}^{1} \frac{1}{3|t^{1/3}|}\sqrt{9t^{2/3}+4}\,dt \quad \overset{\text{FUNZIONE}}{\text{PARI}} \Rightarrow 2\int_{0}^{1}\frac{1}{3t^{1/3}}\sqrt{9t^{2/3}+4}\,dt = \frac{2}{3}\int_{0}^{1} t^{-1/3}\sqrt{9t^{2/3}+4}\,dt =$$

$$d\,9t^{2/3} = \frac{2}{\cancel{3}}\cdot\overset{3}{\cancel{9}}t^{-\frac{1}{3}} = 6t^{-1/3} \qquad = \frac{2}{18}\int_{0}^{1} 6t^{-1/3}\sqrt{9t^{2/3}+4}\,dt = \left|\frac{1}{9}\,\frac{(9t^{2/3}+4)^{\frac{1}{2}+1}}{1+\frac{1}{2}}\right|_{0}^{1} =$$

$$= \frac{2}{3}\cdot\frac{1}{9}\cdot\left|(9t^{2/3}+4)^{3/2}\right|_{0}^{1} = \frac{2}{27}\left((9+4)^{3/2}-(0+4)^{3/2}\right) = \frac{2}{27}\left(13^{3/2}-2^{2\cdot\frac{3}{2}}\right) = \frac{2}{27}\left(13^{3/2}-8\right)$$

13/03/08

① Sia $\varphi:[-2\pi,\pi]\to\mathbb{R}^2$ la curva $\varphi(t)=(x(t),y(t))$ definita da:

$$\begin{cases} x(t)=t \\ y(t)=\cos t \end{cases} t\in[-2\pi,0] \qquad \begin{cases} x(t)=-2t \\ y(t)=1+2t \end{cases} t\in\left[0,\frac{\pi}{2}\right] \qquad \begin{cases} x(t)=-\pi+\pi\cos t \\ y(t)=1+\pi\,\text{sen}\,t \end{cases} t\in\left[\frac{\pi}{2},\pi\right]$$

- disegnare il sostegno di φ, specificando il verso di percorrenza, il punto iniziale e finale, l'equazione (cartesiana) di ciascuno dei 3 tratti.
- scrivere l'equazione parametrica e l'equazione cartesiana della retta tangente alla curva nel punto corrispondente a $t=-\frac{5}{6}\pi$

② Sia $E=\left\{(x,y)\in\mathbb{R}^2:\ -\frac{1}{3}(x-3)^2\le y\le -x+3,\ x^2+y^2\le 9\right\}$

- disegnare E
- scrivere una parametrizzazione, orientata in verso antiorario, di ogni tratto del bordo di E

③ Data la curva $\varphi:[0,3]\to\mathbb{R}^2$ definita da

$$\begin{cases} x(t)=\frac{\sqrt{3}}{2}t \\ y(t)=\frac{1}{2}t \end{cases} t\in[0,1] \qquad \begin{cases} x(t)=\frac{\sqrt{3}}{2} \\ y(t)=\frac{3}{2}-t \end{cases} t\in\,]1,2] \qquad \begin{cases} x(t)=\frac{3\sqrt{3}}{2}-\frac{\sqrt{3}}{2}t \\ y(t)=\frac{1}{2}t-\frac{3}{2} \end{cases} t\in\,]2,3]$$

calcolare gli integrali $\int_\varphi x\,ds$ e $\int_\varphi y\,ds$

④ Data la curva $\rho=4(\text{sen}\,\theta+\cos\theta)$, $\theta\in\left[-\frac{\pi}{4},\frac{3}{4}\pi\right]$

- determinare la retta tangente alla curva nel piano (θ,ρ) in $P\left(\frac{\pi}{2},4\right)$
- dopo aver determinato le eq. cartesiane della curva, disegnate il sostegno nel piano (x,y)

- determinare la tangente alla curva nel piano (x,y) in P(0,4)
- che relazione esiste tra le rette calcolate nei punti 1 e 3?

SOLUZIONI

① a)

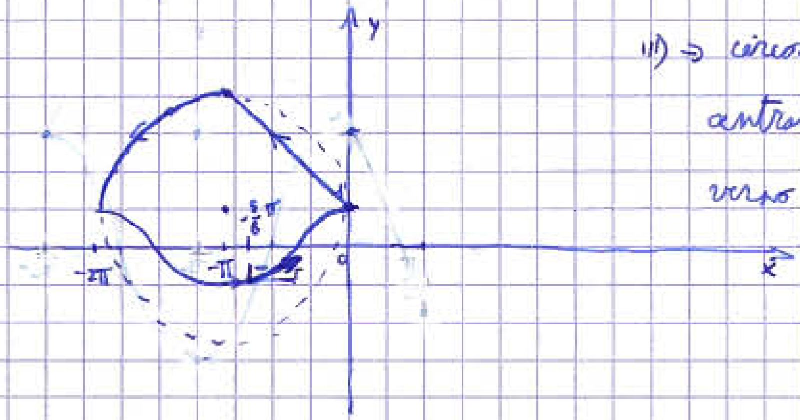
1) $y = \cos x$

11) $y = 1 - x$

111) $(x+\pi)^2 + (y-1)^2 =$

$\pi^2\cos^2 t + \pi^2 \operatorname{sen}^2 t$

$(x+\pi)^2 + (y-1)^2 = \pi^2$



111) ⇒ circonferenza di raggio π centrata in $C(-\pi, 1)$ verso antiorario

$\varphi(-2\pi) = (-2\pi, 1)$ $\quad \varphi(0) = (0,1)$ $\quad \varphi\left(\frac{\pi}{2}\right) = (-\pi, 1+\pi)$ $\quad \varphi(\pi) = (-2\pi, 1)$

Il verso di percorrenza è antiorario come quello della circonferenza

$P_0 = \varphi(-2\pi) = (-2\pi, 1)$ $\quad P_F = \varphi(\pi) = (-2\pi, 1)$ ⇒ la curva è chiusa.

EQUAZIONI CARTESIANE

1) $y = \cos x \quad x \in [-2\pi, 0]$

11) $y = 1 - x \quad x \in [-\pi, 0]$

111) $(x+\pi)^2 + (y-1)^2 = \pi^2 \quad x \in [-2\pi, -\pi]$

b) $\varphi' : \begin{cases} x'(t) = 1 \\ y'(t) = -\operatorname{sen}(t) \end{cases}$ $\quad \varphi'\left(-\frac{5}{6}\pi\right) = \left(1, +\frac{1}{2}\right)$ $\quad \varphi\left(-\frac{5}{6}\pi\right) = \left(-\frac{5}{6}\pi, -\frac{\sqrt{3}}{2}\right)$

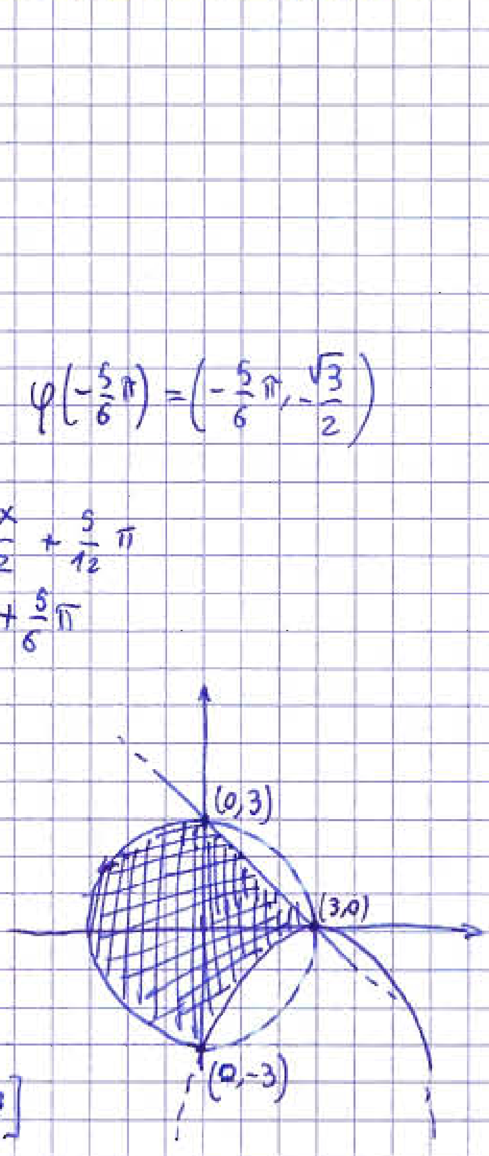
$\begin{cases} x(t) = -\frac{5}{6}\pi + t \\ y(t) = -\frac{\sqrt{3}}{2} + \frac{1}{2}t \end{cases}$ $\quad t = x + \frac{5}{6}\pi$ $\quad y = -\frac{\sqrt{3}}{2} + \frac{x}{2} + \frac{5}{12}\pi$

$2y = x - \sqrt{3} + \frac{5}{6}\pi$

②
$y = -\frac{1}{3}(x-3)^2 = -\frac{1}{3}(x^2 - 6x + 9) = -\frac{1}{3}x^2 + 2x - 3$

$y = -x + 3$

$x^2 + y^2 = 9$ $\quad \begin{cases} x = 3\cos t \\ y = 3\operatorname{sen} t \end{cases}$ $\quad t \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$



$\begin{cases} x = -t \\ y = t + 3 \end{cases}$ $\quad t \in [-3, 0]$ $\quad \begin{cases} x = t \\ y = -\frac{1}{3}(t-3)^2 \end{cases}$ $\quad t \in [0, 3]$

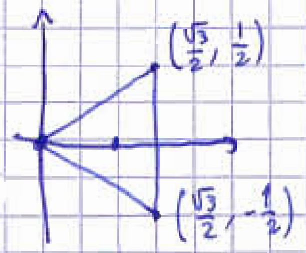
$\varphi = t(0,3) + (1+t)(3,0)$

③ $$\varphi'(t) = \begin{cases} \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) & t \in [0,1] \\ (0,-1) & t \in]1,2] \\ \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) & t \in]2,3] \end{cases} \qquad |\varphi'(t)| = \begin{cases} 1 & t \in [0,1] \\ 1 & t \in]1,2] \\ 1 & t \in]2,3] \end{cases}$$

$$\int_\varphi x\, ds = \int_0^1 \frac{\sqrt{3}}{2} t \cdot 1\, dt + \int_1^2 \frac{\sqrt{3}}{2} \cdot 1\, dt + \int_2^3 \left(3\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} t\right) \cdot 1\, dt = \ldots = \sqrt{3}$$

$$\int_\varphi y\, ds = \int_0^1 \frac{1}{2} t \cdot 1\, dt + \int_1^2 \left(\frac{3}{2} - t\right) \cdot 1\, dt + \int_2^3 \left(\frac{1}{2} t - \frac{3}{2}\right) \cdot 1\, dt = \ldots = 0$$

OPPURE



$$B\left(\frac{2}{3} \cdot \frac{\sqrt{3}}{2}, 0\right) = \left(\frac{\sqrt{3}}{3}, 0\right) \qquad \frac{\sqrt{3}}{3} = \frac{\int_\varphi x\, ds}{\int_\varphi ds} = \frac{\int_\varphi x\, ds}{3} \Rightarrow \int_\varphi x\, ds = 3$$

$$0 = \frac{\int_\varphi y\, ds}{\int_\varphi ds} \Rightarrow \int_\varphi y\, ds = 0$$

④ $$\rho = 4(\cos\theta + \operatorname{sen}\theta) \qquad \theta \in \left[-\frac{\pi}{4}, \frac{3}{4}\pi\right]$$

$$\rho'(\theta) = 4(-\operatorname{sen}\theta + \cos\theta)$$

$$\rho'\left(\frac{\pi}{2}\right) = -4 \qquad \rho\left(\frac{\pi}{2}\right) = 4 \qquad \rho = \rho\left(\frac{\pi}{2}\right) + 4\left(\theta - \frac{\pi}{2}\right) = 4 - 2\pi + 4\theta > 0$$

$$4\theta > 2\pi - 4 \qquad \theta > \frac{\pi}{2} - 1 \qquad \rho = 4 - 2\pi + 4\theta \qquad \theta > \frac{\pi}{2} - 1$$

$$\begin{cases} x(\theta) = 4(\cos\theta + \operatorname{sen}\theta)\cos\theta \\ y(\theta) = 4(\cos\theta + \operatorname{sen}\theta)\operatorname{sen}\theta \end{cases} \qquad \theta \in \left[-\frac{\pi}{4}, \frac{3}{4}\pi\right]$$

$$\begin{cases} x(t) = 4\cos^2 t + 4\operatorname{sen}t\cos t \\ y(t) = 4\operatorname{sen}t\cos t + 4\operatorname{sen}^2 t \end{cases} \qquad t \in \left[-\frac{\pi}{4}, \frac{3}{4}\pi\right]$$

$$x^2 + y^2 = 16\left[\cos^4 t + \operatorname{sen}^2 t\cos^2 t + 2\operatorname{sen}t\cos^3 t + \operatorname{sen}^2 t\cos^2 t + \operatorname{sen}^4 t + 2\operatorname{sen}^3 t\cos t\right] \Rightarrow$$

$$x^2 + y^2 = 16\left[(\cos^2 t + \operatorname{sen}^2 t)^2 + 2\operatorname{sen}t\cos t(\cos^2 t + \operatorname{sen}^2 t)\right] \Rightarrow x^2 + y^2 = 16(1 + 2\operatorname{sen}t\cos t)$$

$$x^2 + y^2 = 16(\cos^2 t + \operatorname{sen}^2 t + 2\operatorname{sen}t\cos t) \qquad x^2 + y^2 = 4(4\cos^2 t + 4\operatorname{sen}^2 t + 8\operatorname{sen}t\cos t)$$

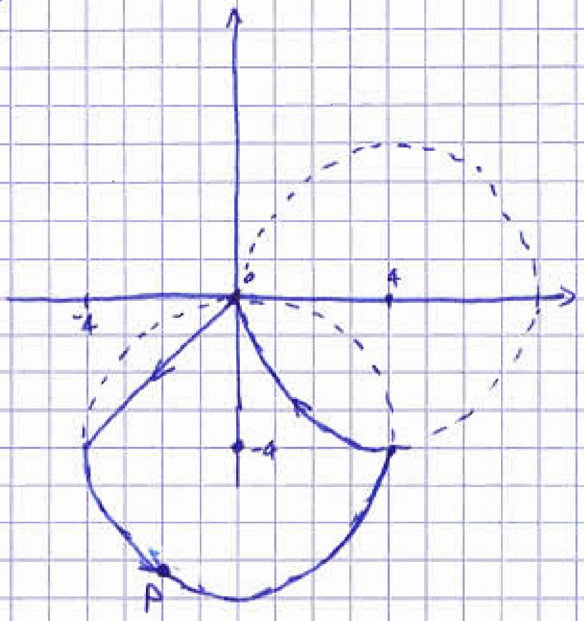
$$x^2 + y^2 = 4x + 4y \qquad (x-2)^2 + (y-2)^2 = 8 \qquad \begin{cases} x(t) = 2 + 2\sqrt{2}\cos t \\ y(t) = 2 + 2\sqrt{2}\operatorname{sen}t \end{cases} \qquad t \in [0, 2\pi]$$

ES. 1.8.12.

$\varphi : [0,6] \to \mathbb{R}^2$

i) $\begin{cases} x=-t \\ y=-t \end{cases}$ $t \in [0,4]$ ii) $\begin{cases} x=-4\cos(\pi t) \\ y=-4-4\,\text{sen}(\pi t) \end{cases}$ $t \in]4,5]$ iii) $\begin{cases} x=4+4\cos\left(\frac{\pi}{2}t\right) \\ y=-4\,\text{sen}\left(\frac{\pi}{2}t\right) \end{cases}$ $t \in]5,6]$

$P\left(-2, -4-2\sqrt{3}\right)$ tg?   i) $y=x$ $t=0$ $\begin{cases} x=0 \\ y=0 \end{cases}$ $t=4$ $\begin{cases} x=4 \\ y=4 \end{cases}$



ii) circonferenza con $C(0,-4)$ e raggio 4
verso antiorario

$t=4$ $\begin{cases} x=-4 \\ y=-4 \end{cases}$ $t=5$ $\begin{cases} x=4 \\ y=-4 \end{cases}$

iii) cfr con $C(4,0)$, raggio 4 e verso orario

$t=5$ $\begin{cases} x=4 \\ y=-4 \end{cases}$ $t=6$ $\begin{cases} x=0 \\ y=0 \end{cases}$

Verso antiorario!

$\begin{cases} -2=-4\cos(\pi t) \\ \cancel{-4}-2\sqrt{3}=\cancel{-4}-4\,\text{sen}(\pi t) \end{cases}$ $\begin{cases} \cos(\pi t)=\frac{1}{2} \\ \text{sen}(\pi t)=\frac{\sqrt{3}}{2} \end{cases}$ $\pi t=\frac{\pi}{3}$ $t=\frac{1}{3}$ ∉ all'intervallo ✗

$\pi t=4\pi+\frac{\pi}{3}$ $t=\frac{13}{3}$ e all'intervallo!

φ' $\begin{cases} x=4\,\text{sen}(\pi t)\cdot\pi \\ y=-4\cos(\pi t)\cdot\pi \end{cases}$ $\begin{cases} x=4\pi\cdot\frac{\sqrt{3}}{2}=2\sqrt{3}\pi \\ y=-4\pi\cdot\frac{1}{2}=-2\pi \end{cases}$ $\varphi'(P)=\left(2\sqrt{3}\pi ; -2\pi\right)$

$\frac{x-x_P}{x'}=\frac{y-y_P}{y'}$ $\frac{x+2}{2\sqrt{3}\pi}=\frac{y+4+2\sqrt{3}}{-2\pi}$ $x+2=-\sqrt{3}y-4\sqrt{3}-6$ $\sqrt{3}y=-x-4\sqrt{3}-8$

ES 1.8.13

$\varphi : [-1,5] \to \mathbb{R}^2$ $\begin{cases} x=-3t^2-6t \\ y=3t+3 \end{cases}$ $t \in [-1,0[$ $\begin{cases} x=-4+4\cos\left(\frac{\pi}{2}t\right) \\ y=3-3\,\text{sen}\left(\frac{\pi}{2}t\right) \end{cases}$ $t \in [0,1[$

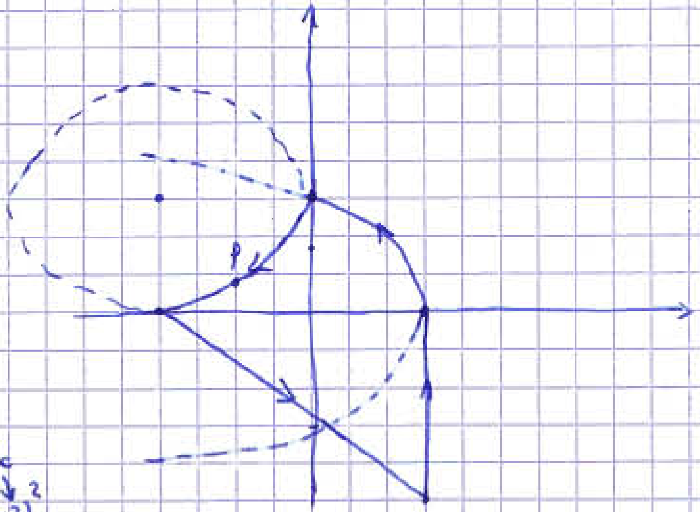
$\begin{cases} x=\frac{7}{3}t-\frac{19}{3} \\ y=-\frac{5}{3}t+\frac{5}{3} \end{cases}$ $t \in [1,4]$ $\begin{cases} x=3 \\ y=5t-25 \end{cases}$ $t \in]4,5]$ $P\left(-2; \frac{3}{2}\left(2-\sqrt{3}\right)\right)$

I) $\begin{cases} x = -3\left(\frac{y}{3}-1\right)^2 - 6\left(\frac{y}{3}-1\right) \\ t = \frac{y}{3} - 1 \end{cases}$ $x = -3\left(\frac{y^2}{9} + 1 - \frac{2}{3}y\right) - 2y + 6$ parabola

$x = -\frac{y^2}{3} - 3 + \cancel{2y} - \cancel{2y} + 6 \quad x = -\frac{y^2}{3} + 3$

$V = (3;0)$

$t = -1 \quad \begin{cases} x = 3 \\ y = 0 \end{cases} \qquad t = 0 \quad \begin{cases} x = 0 \\ y = 3 \end{cases}$



II) Ellisse

$C(-4,3) \quad a = 4 \quad b = 3$ verso orario

$t = 0 \begin{cases} x = 0 \\ y = 3 \end{cases} \qquad t = 1 \begin{cases} x = -4 \\ y = 0 \end{cases} \qquad \frac{(x \overset{-x_c}{+4})^2}{\underset{a^2 \rightarrow}{16}} + \frac{(y \overset{-y_c}{-3})^2}{\underset{\leftarrow b^2}{9}} = 1$

III) $\begin{cases} 3x = 7t - 19 \\ y = -\frac{5}{3}t + \frac{5}{3} \end{cases} \qquad \begin{cases} t = \frac{3x+19}{7} \\ y = -\frac{5}{3} \cdot \frac{3x+19}{7} + \frac{5}{3} \end{cases}$

$t = 1 \quad P(-4,0)$

$t = 4 \quad P(3,-5)$

IV) $x = 3$ retta verticale $\quad t = 4 \quad P(3,-5) \quad t = 5 \quad P(3,0)$

Tangente → 2° tratto

$\begin{cases} -2 = -4 + 4\cos\left(\frac{\pi}{2}t\right) \\ \cancel{3} - \frac{3}{2}\sqrt{3} = \cancel{3} - 3\,\mathrm{sen}\left(\frac{\pi}{2}t\right) \end{cases} \qquad \begin{cases} \cos\left(\frac{\pi}{2}t\right) = \frac{1}{2} \\ \mathrm{sen}\left(\frac{\pi}{2}t\right) = \frac{\sqrt{3}}{2} \end{cases}$

$x' = -\overset{2}{\cancel{4}}\,\mathrm{sen}\left(\frac{\pi}{2}t\right) \cdot \frac{\pi}{\cancel{2}}$

$y' = -3\cos\left(\frac{\pi}{2}t\right) \cdot \frac{\pi}{2}$

$\varphi'(P) = \left(-\sqrt{3}\pi; -\frac{3}{4}\pi\right)$

EQUAZIONE PARAMETRICA

$\begin{cases} x = -2 - \sqrt{3}\pi t \\ y = 3 - \frac{3}{2}\sqrt{3} - \frac{3}{4}\pi t \end{cases}$

EQUAZIONE CARTESIANA

$\frac{x+2}{-\sqrt{3}\cancel{\pi}} = \frac{y - 3 + \frac{3}{2}\sqrt{3}}{-\frac{3}{4}\cancel{\pi}} \qquad \frac{-x-2}{\cancel{\sqrt{3}}} = \left(y - 3 + \frac{3}{2}\sqrt{3}\right) \cdot \left(-\frac{4}{\sqrt{3}}\right)$

$\sqrt{3}x + 2\sqrt{3} = 4y - 12 + 6\sqrt{3} \qquad \sqrt{3}x - 4y = 4\sqrt{3} - 12$

ES 1.8.19

$\varphi : [0, \pi] \to \mathbb{R}^2$

$\varphi(t) = (e^t sent, e^t cost)$ $t \in [0, \pi]$ tg in $\varphi(0), \varphi(\frac{\pi}{2}), \varphi(\pi)$ Lunghezza e baricentr

$$\begin{cases} x = e^t sent \\ y = e^t cost \end{cases} \quad t \in [0,\pi] \qquad \begin{cases} x' = e^t sent + e^t cost = e^t(sent + cost) \\ y' = e^t cost - e^t sent = e^t(cost - sent) \end{cases}$$

$\varphi'(0) = (1, 1)$ $\varphi'(\frac{\pi}{2}) = (e^{\pi/2}, -e^{\pi/2})$ $\varphi'(\pi) = (-e^\pi, -e^\pi)$

$P(0,1)$ $P(e^{\pi/2}, 0)$ $P = (0, -e^\pi)$

$$\frac{x-0}{1} = \frac{y-1}{1} \qquad \frac{x - e^{\pi/2}}{e^{\pi/2}} = \frac{y-0}{-e^{\pi/2}} \qquad \frac{x-0}{\cancel{-}e^\pi} = \frac{y+e^\pi}{-e^\pi}$$

$x = y - 1$ $y = x+1$ $x - e^{\pi/2} = -y$ $y = -x + e^{\pi/2}$ $y = x - e^\pi$

$$\mathcal{L} = \int_0^\pi \sqrt{(x')^2 + (y')^2} = \int_0^\pi \sqrt{e^{2t}(\underbrace{sen^2t + cos^2t}_{1} + 2\,sent\,cost) + e^{2t}(\underbrace{cos^2t + sen^2t}_{1} - 2\,sent\,cost)}\,dt =$$

$$= \int_0^\pi \sqrt{e^{2t}(1 + \cancel{2sentcost} + 1 - \cancel{2sentcost})}\,dt = \int_0^\pi \underbrace{\sqrt{2e^{2t}}\,dt}_{ds} = \int_0^\pi e^t \cdot \sqrt{2}\,dt = \left[\sqrt{2}\,e^t\right]_0^\pi = \sqrt{2}(e^\pi - 1)$$

BARICENTRO

$$x = \frac{1}{L} \cdot \int_\varphi x\,ds = \frac{1}{\sqrt{2}(e^\pi - 1)} \int_0^\pi e^t sent \cdot \sqrt{2e^{2t}}\,dt = \frac{1}{\sqrt{2}(e^\pi-1)} \int_0^\pi e^t \cdot sent \cdot e^t \cancel{\sqrt{2}}\,dt =$$

$$= \int_0^\pi e^{2t} sent\,dt \cdot \frac{1}{e^\pi - 1} = \frac{1}{e^\pi - 1} \cdot \left| \frac{-e^{2t}cost + 2e^{2t}sent}{5} \right|_0^\pi = \frac{1}{e^\pi - 1}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right)$$

$$\left[\int_0^\pi \underset{f}{e^{2t}}\,\underset{g'}{sent}\,dt = -e^{2t}cost + \int 2e^{2t}(cost)\,dt = -e^{2t}cost + 2\left(e^{2t}sent - \int 2e^{2t} \cdot sent\,dt\right) = \right.$$

$$= -e^{2t}cost + 2e^{2t}sent - 4\int e^{2t}sent\,dt \qquad 5\int e^{2t}sent\,dt = e^{2t}(-cost + 2sent)$$

$$\int e^{2t}sent\,dt = \frac{e^{2t}}{5}(-cost + 2sent)$$

1.8.22

$\sqrt{(x')^2+(y')^2}$ o $|\varphi'(t)|$

$$\int_{\varphi} 5xy^2\,ds \qquad \varphi(t)=\left(\frac{t^5}{5},t\right) \quad t\in[0,1] \qquad \varphi'(t)=\left(t^4,1\right)$$

$$\int_0^1 8\cdot\frac{t^5}{8}\cdot t^2\cdot\sqrt{(t^4)^2+(1)^2}\,dt=\int_0^1 t^7\sqrt{t^8+1}\,dt=\frac{1}{8}\int_0^1 8t^7\cdot(t^8+1)^{1/2}\,dt=$$

$$=\frac{1}{8}\left[\frac{(t^8+1)^{3/2}}{3/2}\right]_0^1=\frac{1}{\underset{4}{8}}\left[\frac{2}{3}\sqrt{(t^8+1)^3}\right]_0^1=\frac{1}{12}\left(\sqrt{2^3}-1\right)=\frac{1}{12}(2\sqrt{2}-1)$$

1.8.24

$$\rho=-8\cos\theta \quad \theta\in\left[\frac{\pi}{2},\frac{3}{2}\pi\right] \qquad \begin{cases}x=\rho\cos\theta\\ y=\rho\,\text{sen}\,\theta\end{cases}$$

$$\circledast\begin{cases}x=-8\cos\theta\cdot\cos\theta=-8\cos^2\theta\\ y=-8\,\text{sen}\,\theta\cos\theta \quad \text{eq. param.}\end{cases}$$

$$\begin{cases}x=-8\,\dfrac{1+\cos 2\theta}{2}=-4-4\cos 2\theta\\ y=-4\,\text{sen}\,2\theta\end{cases} \qquad 2\theta\in[\pi,3\pi]$$

D
A

R
I
C
O
R
D
A
R
E

$$\text{sen}\,\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

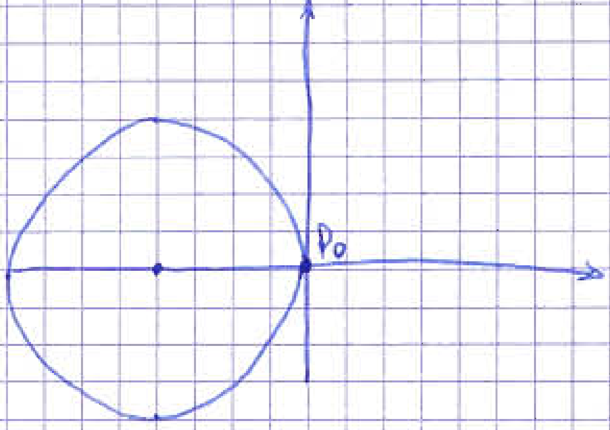
$$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$$

$$\text{sen}^2 x=\frac{1-\cos 2x}{2}$$

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

Circonferenza di raggio 4, C(-4,0) e verso antiorario

$$2\theta=\pi \quad \begin{cases}x=0\\ y=0\end{cases}$$



$\circledast$ oppure sommo i quadrati

$$x^2+y^2=64\cos^4\theta+64\,\text{sen}^2\cos^2\theta \qquad x^2+y^2=64\cos^4\theta+64\cos^2\theta(1-\cos^2\theta)$$

$$x^2+y^2=\cancel{64\cos^4\theta}+64\cos^2\theta-\cancel{64\cos^4\theta} \qquad x^2+y^2=64\cos^2\theta \qquad \left(\frac{x}{8}\right)^2+\left(\frac{y}{8}\right)^2=(\cos\theta)^2$$

$\rho = -4\,\text{sen}\,\theta \quad \theta \in [\pi, 2\pi]$

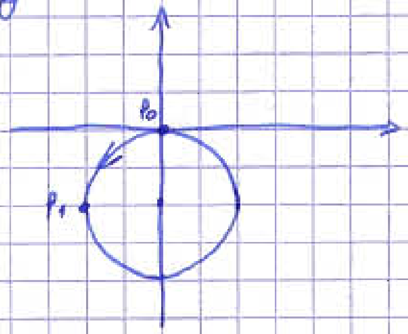
$$\begin{cases} x = -4\,\text{sen}\,\theta\cos\theta = -2\,\text{sen}\,2\theta \\ y = -4\,\text{sen}^2\theta = -\overset{2}{\cancel{4}} \cdot \dfrac{1-\cos 2\theta}{\cancel{2}} = -2+2\cos 2\theta \end{cases}$$

$2\theta \in [2\pi, 4\pi]$

Cfr. $C(0,-2)$ $R=2$ verso antiorario

$2\theta = 2\pi \quad P_0(0,0)$

$2\theta = \frac{5}{2}\pi \quad P_1(-2,-2)$



Es da esame

$E = \{(x y) \in \mathbb{R}^2 : x \leq 2,\ 2x^2-4x+2 \leq y \leq e^x+1\}$

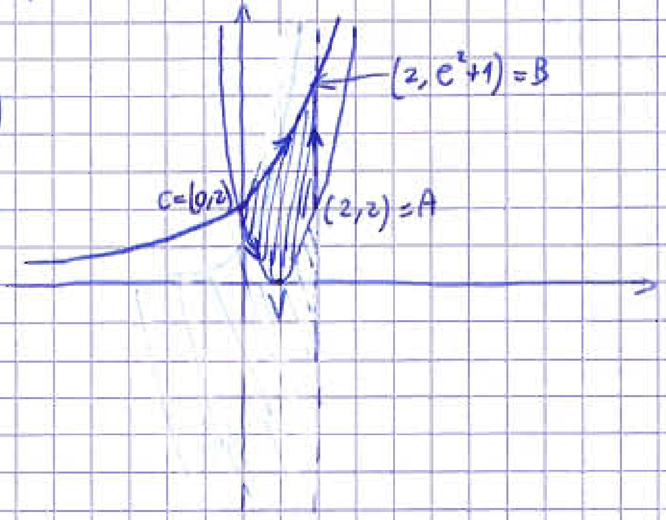
I) Disegnare E

II) Scrivere una parametrizzazione per i bordi

a) $x=2$ retta verticale

b) $y = 2x^2-4x+2$ parabola $V=(1,0)$

c) $y = e^x+1$



$$AB: \begin{cases} x=2 \\ y=t \end{cases} \quad t \in [2, e^2+1]$$

$$CA: \begin{cases} x=t \\ y=2t^2-4t+2 \end{cases} \quad t \in [0, 2[$$

$$BC: \begin{cases} x=t \\ y=e^t+1 \end{cases} \quad t \in [0, 2[$$

ESERCIZI

.8.2.

$\varphi : [2,4] \to \mathbb{R}^2$ $\quad \begin{cases} x(t) = t-4 \\ y(t) = t^2-6t+8 \end{cases} \quad t \in [2,4] \qquad \varphi' : \begin{cases} x'(t) = 1 \\ y'(t) = 2t-6 \end{cases}$

$\varphi'\left(\frac{7}{2}\right) = (1,1) \qquad \varphi\left(\frac{7}{2}\right) = \left(-\frac{1}{2}, -\frac{3}{4}\right)$

1.8.3.

i) $\varphi : \begin{cases} x(t) = |10t-t^2| \\ y(t) = \cos t^2 - 5 \end{cases} \quad t \in [3,4] \quad \begin{matrix} 10t-t^2 > 0 \\ \forall t \in [3,4] \end{matrix} \quad \begin{cases} x(t) = 10t-t^2 \\ y(t) = \cos t^2 - 5 \end{cases} \quad t \in [3,4] \quad \begin{cases} x'(t) = 10-2t \\ y'(t) = -\operatorname{sen} t^2 \cdot 2t \end{cases} \quad t \in [3,4]$

funz. cont. $\Rightarrow \varphi$ di classe C^1

ii) $\varphi : \begin{cases} x(t) = \cos|t| + 5 \\ y(t) = \operatorname{sen}(t^2-3t) \end{cases} \quad t \in [-1,5] \quad \cos t = \cos(-t) \quad \begin{cases} x(t) = \cos t + 5 \\ y(t) = \operatorname{sen}(t^2-3t) \end{cases} \quad t \in [-1,5] \quad \begin{cases} x'(t) = -\operatorname{sen} t \\ y'(t) = \cos(t^2-3t)\cdot(2t-3) \end{cases}$ φ di classe C

iii) $\varphi : \begin{cases} x(t) = |5t-t^2| \\ y(t) = \operatorname{sen}(\pi(t^2-1)) \end{cases} \quad t \in [3,4] \quad \begin{matrix} 5t-t^2 > 0 \\ \forall t \in [3,4] \end{matrix} \quad \begin{cases} x(t) = 5t-t^2 \\ y(t) = \operatorname{sen} \pi(t^2-1) \end{cases} \quad t \in [3,4] \quad \begin{cases} x'(t) = 5-2t \\ y'(t) = \cos \pi(t^2-1)\cdot 2\pi t \end{cases} \quad t \in [3,4]$ φ di classe

iv) $\varphi : \begin{cases} x(t) = \operatorname{sen}|t| - 2 \\ y(t) = \cos t \end{cases} \quad t \in [-1,1]$ non di classe C^1

1.8.4.

i) $\varphi : \begin{cases} x(t) = 2 + \cos 3t \\ y(t) = -5 + \operatorname{sen} 3t \end{cases} \quad t \in \left]0, 4\pi\right]$ φ non chiusa perchè $0 \notin I$

ii) $\varphi : \begin{cases} x(t) = t^3 - 5t^2 + 4t \\ y(t) = t^8 - 6t^4 \end{cases} \quad t \in [0,1] \quad \begin{matrix} \varphi(0) = (0,0) \\ \varphi(1) = (4,0) \end{matrix}$ φ non è chiusa

ii) $\varphi : \begin{cases} x(t) = e^{t^2+8t} \\ y(t) = \operatorname{sen}(\pi t) + 5 \end{cases} \quad t \in [-8,0] \quad \begin{matrix} \varphi(-8) = (1,5) \\ \varphi(0) = (1,5) \end{matrix}$ φ è chiusa

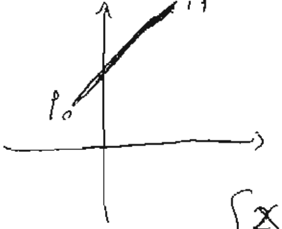
) $\varphi : \begin{cases} x(t) = t^4 - 1 \\ y(t) = 6t^2 - \cos(\pi t) \end{cases} \quad t \in [0,2] \quad \begin{matrix} \varphi(0) = (-1,-1) \\ \varphi(2) = (15, 23 \end{matrix}$ φ non è chiusa

) $\varphi : \begin{cases} x(t) = t^3 - t^6 \\ y(t) = \operatorname{sen}(\pi t) - 6 \end{cases} \quad t \in]0,1]$ φ non è chiusa perchè $0 \notin I$

i) $\varphi : \begin{cases} x(t) = t^2 + t \\ y(t) = \operatorname{sen}(\pi t) - 6 \end{cases} \quad t \in [-1,0] \quad \begin{matrix} \varphi(-1) = (0,-6) \\ \varphi(0) = (0,-6) \end{matrix}$ φ è chiusa

1.8.5

$P_0 = (-1,2) \quad P_1 = (2,5)$



$$\begin{cases} x(t) = -1 + t(2-(-1)) \\ y(t) = 2 + t(5-(+2)) \end{cases} \quad \begin{cases} x(t) = -1 + 3t \\ y(t) = 2 + 3t \end{cases} \text{DA } P_0 \text{ A } P_1 \quad t \in [0,1]$$

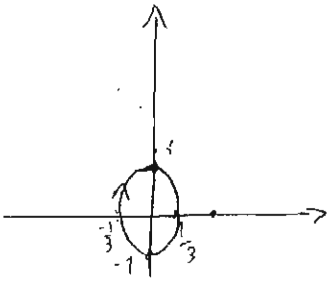
$$\begin{cases} x(t) = 2 + t(-1-2) \\ y(t) = 5 + t(2-5) \end{cases} \quad \begin{cases} x(t) = 2 - 3t \\ y(t) = 5 - 3t \end{cases} \text{DA } P_1 \text{ A } P_0 \quad t \in [0,1]$$

1.8.6.

$x^2 + 9y^2 = 1$

$P_i = (-1,0)$

$P_0 = \left(\frac{\sqrt{3}}{2}, \frac{1}{6}\right)$



$$x^2 + \left(\frac{y}{\frac{1}{3}}\right)^2 = 1$$

$$\begin{cases} x = \cos t \\ y = \frac{1}{3} \operatorname{sen} t \end{cases} \quad t \in [-\pi, \pi]$$

$$\begin{cases} \frac{\sqrt{3}}{2} = \cos t \\ \frac{1}{6} = \frac{1}{3} \operatorname{sen} t \end{cases} \quad \begin{cases} \cos t = \frac{\sqrt{3}}{2} \\ \operatorname{sen} t = \frac{1}{2} \end{cases} \quad t = \frac{\pi}{6} \quad \varphi'(t) = \left(-\operatorname{sen} t, \frac{1}{3}\cos t\right)$$

$$r: \left(\frac{\sqrt{3}}{2}, \frac{1}{6}\right) + t\left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right) \quad r: \begin{cases} x(t) = \frac{\sqrt{3}}{2} - \frac{1}{2}t \\ y(t) = \frac{1}{6} + \frac{\sqrt{3}}{6}t \end{cases} \quad \begin{cases} t = \sqrt{3} - 2x \\ y = \frac{1}{6} + \frac{\sqrt{3}}{6} \cdot (\sqrt{3} - 2x) \end{cases}$$

1.8.7.

I)

$P_1 = (0,1)$

$P_2 = (-1,3)$

$Q = (-1,2)$

$$\frac{y-1}{3-1} = \frac{x\cancel{-0}}{-1\cancel{-0}} \quad y - 1 = -2x \quad y = 1 - 2x \quad \varphi \begin{cases} x(t) = t \\ y(t) = 1 - 2t \end{cases} t \in \mathbb{R}$$

$\varphi'(t) = (1, -2)$

$$T = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) \quad N = \left(\frac{-2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$$

$|\varphi'(t)| = \sqrt{1+4} = \sqrt{5}$

II)

$r = 3$

$C = (0,0)$

verso orario

$P_i\ (0,3)$

$Q\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$

$$\begin{cases} x(t) = -3\cos t \\ y(t) = 3 \operatorname{sen} t \end{cases} \quad t \in \left[+\frac{\pi}{2}, \frac{5}{2}\pi\right] \quad \begin{cases} \frac{3\sqrt{2}}{2} = -3\cos t \\ -\frac{3\sqrt{2}}{2} = 3 \operatorname{sen} t \end{cases} \quad \begin{cases} \cos t = -\frac{\sqrt{2}}{2} \\ \operatorname{sen} t = -\frac{\sqrt{2}}{2} \end{cases} \quad t_0 = \frac{5}{4}\pi$$

$$\varphi'(t) = (3 \operatorname{sen} t, 3\cos t) \quad |\varphi'(t_0)| = \sqrt{9(\operatorname{sen}^2 t + \cos^2 t)} = 3$$

$$\varphi'\left(\frac{5}{4}\pi\right) = \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right) \quad T = \left(\frac{-\frac{3\sqrt{2}}{2}}{3}, \frac{-\frac{3\sqrt{2}}{2}}{3}\right) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \quad N = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

III)

$r = 2$

$C = (-3,-2)$

verso antior.

$P_i = (-3,-4)$

$Q = (-3-\sqrt{3}, -1)$

$$\begin{cases} x(t) = -3 + 2\cos t \\ y(t) = -2 + 2 \operatorname{sen} t \end{cases} \quad t \in \left[-\frac{\pi}{2}, \frac{3}{2}\pi\right] \quad \begin{cases} \cancel{-3} - \sqrt{3} = \cancel{-3} + 2\cos t \\ -1 = -2 + 2 \operatorname{sen} t \end{cases} \quad \begin{cases} \cos t = -\frac{\sqrt{3}}{2} \\ \operatorname{sen} t = \frac{1}{2} \end{cases} \quad t = \frac{5}{6}\pi$$

$$\varphi'(t) = (-2 \operatorname{sen} t, 2\cos t) \quad \varphi'\left(\frac{5}{6}\pi\right) = (-1, -\sqrt{3}) \quad T = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad N = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\left|\varphi'\left(\frac{5}{6}\pi\right)\right| = \sqrt{1+3} = 2$$

V) ellisse
centro (1,1)
semiassi 2, 1
verso antior.
Pi (3,1)

$$\begin{cases} x(t) = 1 + 2\cos t \\ y(t) = 1 + \operatorname{sen} t \end{cases} \quad t \in [0, 2\pi] \qquad \begin{cases} 1+\sqrt{3} = 1 + 2\cos t \\ \frac{3}{2} = 1 + \operatorname{sen} t \end{cases} \quad \begin{cases} \cos t = \frac{\sqrt{3}}{2} \\ \operatorname{sen} t = \frac{1}{2} \end{cases} \quad t = \frac{\pi}{6}$$

(2

$$z = \left(1+\sqrt{3}, \frac{3}{2}\right) \qquad \varphi' = (-2\operatorname{sen} t, \cos t) \qquad \varphi'\left(\frac{\pi}{6}\right) = \left(-1, \frac{\sqrt{3}}{2}\right) \qquad \left|\varphi'\left(\frac{\pi}{6}\right)\right| = \sqrt{1+\frac{3}{4}} = \frac{\sqrt{7}}{2}$$

$$T = \left(\frac{-1}{\frac{\sqrt{7}}{2}}, \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{7}}{2}}\right) = \left(-\frac{2}{\sqrt{7}}, \frac{\sqrt{3}}{\sqrt{7}}\right) \qquad N = \left(\frac{\sqrt{3}}{\sqrt{7}}, \frac{2}{\sqrt{7}}\right)$$

1.8.8.

$$\varphi(t) = (2\cos t, \operatorname{sen} 2t) = (2\cos t, 2\operatorname{sen} t\cos t) \qquad t \in [0, 2\pi]$$

$$\varphi'(t) = (-2\operatorname{sen} t, 2\cos 2t) \qquad \begin{cases} -2\operatorname{sen} t = 0 \\ 2\cos 2t = 0 \end{cases} \quad \begin{cases} \operatorname{sen} t = 0 \\ 2 - 2\operatorname{sen}^2 t = 0 \end{cases} \quad \text{MAI} \Rightarrow \varphi \text{ è Regolare}$$

$$\begin{cases} 2\cos t_1 = 2\cos t_2 \\ \operatorname{sen} 2t_1 = \operatorname{sen} 2t_2 \end{cases} \quad \begin{cases} \cos t_1 = \cos t_2 \\ 2\operatorname{sen} t_1 \cos t_1 = 2\operatorname{sen} t_2 \cos t_2 \end{cases} \quad \begin{cases} \ldots \\ 2\operatorname{sen} t_1 \cos t_2 = 2\operatorname{sen} t_2 \cos t_2 \end{cases}$$

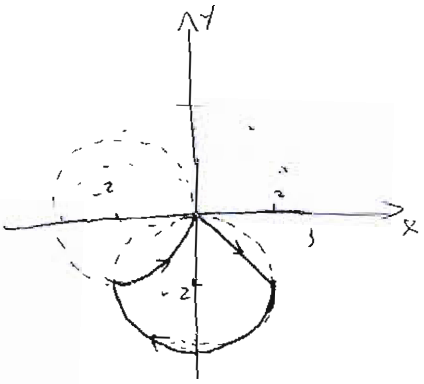
$\cos t_2 = 0$ se $t_2 = \frac{\pi}{2}$ o $\frac{3}{2}\pi$, se $t_1 = \frac{\pi}{2}$ e $t_2 = \frac{3}{2}\pi$ φ non è semplice.

$\varphi(0) = (2,0)$ $\quad \varphi(2\pi) = (2,0)$ $\quad \varphi$ è chiusa.

$$\varphi'(0) = (0,2) \qquad \varphi'\left(\frac{\pi}{2}\right) = (-2,-2) \qquad \varphi'(\pi) = (0,2) \qquad \varphi'\left(\frac{3}{2}\pi\right) = (2,-2)$$

1.8.10

$$\begin{cases} x(t) = t \\ y(t) = -t \end{cases} \quad t \in [0,2] \qquad |\varphi'(t)| = \sqrt{1+1} = \sqrt{2}$$



$$\begin{cases} x(t) = -2\cos\left(\frac{\pi}{2}t\right) \\ y(t) = -2 + 2\operatorname{sen}\left(\frac{\pi}{2}t\right) \end{cases} \quad t \in]2,4] \qquad |\varphi'(t)| = \sqrt{\left(\frac{\pi}{2}\cdot\operatorname{sen}\frac{\pi}{2}t\right)^2 + \left(2\cos\frac{\pi}{2}t\cdot\frac{\pi}{2}\right)^2} =$$

$$|\varphi'(t)| = \sqrt{\left[2\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\right]^2 + \left[2\frac{\pi}{2}\operatorname{sen}\left(\frac{\pi}{2}t\right)\right]^2} =$$

$$\begin{cases} x(t) = -2 + 2\operatorname{sen}\left(\frac{\pi}{2}t\right) \\ y(t) = -2\cos\left(\frac{\pi}{2}t\right) \end{cases} \quad t \in]4,5] \qquad \rightarrow x^2 + (y-2)^2 = 4 \quad x \in [-2, 2]$$

Verso orario

$$\rightarrow (x+2)^2 + y^2 = 4 \quad x \in [-2, 0]$$

$$\mathcal{L} = \int_\varphi |\varphi'(t)|\,dt = \int_0^2 \sqrt{2}\,dt + \int_2^4 \pi\,dt + \int_4^5 \pi\,dt = \left[\sqrt{2}t\right]_0^2 + \left[\pi t\right]_2^4 + \left[\pi t\right]_4^5 = 2\sqrt{2} + \cancel{4\pi} - 2\pi + 5\pi - \cancel{4\pi} =$$

$$= 2\sqrt{2} + 3\pi$$

Calcolabile anche con le formule della geometria elementare

ESERCITAZIONE

27/03/2008

①

① Sia $\varphi : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ la curva $\varphi(t) = (x(t), y(t)) = (\cos^3 t, \operatorname{sen}^3 t)$, $t \in [0, \frac{\pi}{2}]$

- Dite se la curva è di classe C^1, regolare, semplice, chiusa
- Scrivete l'equazione parametrica e l'equazione cartesiana della retta tangente alla curva nel punto $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$
- Calcolate $\mathcal{L}(\varphi)$, la lunghezza di φ.
- Data la funzione $f(x,y) = x+y$, calcolate l'integrale curvilineo $\int_\varphi f\, ds$
- Scrivete l'equazione cartesiana del sostegno di φ, specificando punto iniziale e punto finale.

② Date le funzioni $f(x,y) = |x| + |y|$ e $g(x,y) = x^2 + (y-2)^2$

- disegnate gli insiemi $\{f \leq 4\}$ e $\{g \geq 4\}$
- parametrizzate in verso orario ciascun tratto del bordo dell'insieme $\{f \leq 4\} \cap \{g \geq 4\}$

③ Data la funzione $f(x,y) = (y-x-3)\log(y+2x^2)$

- determinate il dominio massimale $\operatorname{dom}(f)$ della funzione
- disegnate gli insiemi $\{f=0\}$, $\{f \leq 0\}$, $\{f \geq 0\}$

④ Data la funzione $f(x,y) = xy$

- disegnate gli insiemi $\{f=1\}$, $\{f \leq 1\}$, $\{f \geq 1\}$, $\{f=0\}$, $\{f \leq 0\}$, $\{f \geq 0\}$, $\{f=-1\}$, $\{f \leq -1$ $\{f \geq -1\}$
- attraverso lo studio degli insiemi di livello, determinate i punti di massim e minimo assoluti di f sul triangolo di vertici $(-1,3)$, $(2,0)$, $(-1,-3)$

① $\varphi'(t)=(3\cos^2 t\cdot(-\text{sen}\,t),\ 3\cos t\,\text{sen}^2 t)=(-3\,\text{sen}\,t\cos^2 t,\ 3\,\text{sen}^2 t\cos t)$

$\varphi'(t)$ è continua $\Rightarrow$ φ è di classe C^1.

$$\begin{cases}-3\,\text{sen}\,t\cos^2 t=0\\ 3\,\text{sen}^2 t\cos t=0\end{cases}\qquad \begin{matrix}\text{sen}\,t=0 \ \text{o} \ \cos^2 t=0\\ \text{sen}^2 t=0 \ \text{o} \ \cos t=0\end{matrix}\qquad \begin{matrix}t=0 \ \text{o} \ t=\frac{\pi}{2}\\ t=0 \ \text{o} \ t=\frac{\pi}{2}\end{matrix}\quad \notin\ ]0,\tfrac{\pi}{2}[$$

φ è regolare $\quad \varphi'(t)\neq(0,0)\quad \forall t\in\ ]0,\frac{\pi}{2}[$

φ è semplice perchè $\cos^3 t$ e $\text{sen}^3 t$ sono iniettive tra 0 e $\frac{\pi}{2}$.

$\varphi(0)=(1,0)\qquad \varphi(\frac{\pi}{2})=(0,1)\qquad \varphi$ non è chiusa

$$\begin{cases}\frac{\sqrt2}{4}=\cos^3 t\\ \frac{\sqrt2}{4}=\text{sen}^3 t\end{cases}\qquad \begin{cases}\cos t=\sqrt[3]{\frac{\sqrt2}{4}}=\frac{\sqrt[6]{2}}{\sqrt[3]{4}}=\frac{\sqrt[6]{2}}{\sqrt2\cdot\sqrt[6]{2}}=\frac{1}{\sqrt2}=\frac{\sqrt2}{2}\\ \text{sen}\,t=\sqrt[3]{\frac{\sqrt2}{4}}=\frac{\sqrt[6]{2}}{\sqrt[3]{4}}=\frac{\sqrt[6]{2}}{\sqrt2\cdot\sqrt[6]{2}}=\frac{1}{\sqrt2}=\frac{\sqrt2}{2}\end{cases}\qquad t=\frac{\pi}{4}$$

$$\begin{cases}x'=-3\,\text{sen}\,t\cos^2 t\\ y'=3\,\text{sen}^2 t\cos t\end{cases}\qquad \varphi'(\tfrac{\pi}{4})=\left(-\frac{3\sqrt2}{4},\ \frac{3\sqrt2}{4}\right)\qquad \frac{x-\frac{\sqrt2}{4}}{-\frac{3\sqrt2}{4}}=\frac{y-\frac{\sqrt2}{4}}{\frac{3\sqrt2}{4}}\ \text{eq. cart.}$$

$$\begin{cases}x=\frac{\sqrt2}{4}-\frac{3\sqrt2}{4}t\\ y=\frac{\sqrt2}{4}+\frac{3\sqrt2}{4}t\end{cases}\ \text{eq. param.}\qquad |\varphi'(t)|=\sqrt{9\,\text{sen}^2 t\cos^4 t+9\,\text{sen}^4 t\cos^2 t}=$$

$$=\sqrt{9\,\text{sen}^2 t\cos^2 t\underbrace{(\text{sen}^2 t+\cos^2 t)}_{1}}=3\,\text{sen}\,t\cos t$$

~~$\int_0^{\pi/2}\frac{3}{2}dt=\left[\frac{3}{2}t\right]_0^{\pi/2}=\frac{3}{4}\pi+0=\frac{3}{4}\pi$~~

$$\mathcal{L}(\varphi)=\int_0^{\pi/2}3\,\text{sen}\,t\cos t\,dt=3\cdot\frac{1}{2}\int_0^{\pi/2}2\,\text{sen}\,t\cos t\,dt=\frac{3}{2}\left[\text{sen}^2 t\right]_0^{\pi/2}=\frac{3}{2}$$

$f=x+y$

$$\int_\varphi f\,ds=\int_0^{\pi/2}(\cos^3 t+\text{sen}^3 t)\cdot3\,\text{sen}\,t\cos t\,dt=3\int_0^{\pi/2}(\text{sen}\,t\cos^4 t+\text{sen}^4 t\cos t)\,dt=3\left(\int_0^{\pi/2}\text{sen}\,t\cos^4 t\,dt+\int_0^{\pi/2}\text{sen}^4 t\cos t\right.$$

$$=3\left[-\frac{\cos^5 t}{5}+\frac{\text{sen}^5 t}{5}\right]_0^{\pi/2}=3\left[-0+\frac{1}{5}-\left(-\frac{1}{5}+0\right)\right]=3\left(\frac{2}{5}\right)=\frac{6}{5}$$

$$\begin{cases}x(t)=\cos^3 t\\ y(t)=\text{sen}^3 t\end{cases}\qquad \varphi(0)=(1,0);\ \varphi(\tfrac{\pi}{2})=(0,1)\qquad x^{1/3}(t)=\cos t\ \text{e}\ y^{1/3}(t)=\text{sen}\,t$$

$$x^{2/3}+y^{2/3}=1$$

②

27/03/08 ②

f) $|x| + |y| = 4 \qquad |y| = 4 - |x|$

$|y| \leq 4 - |x|$

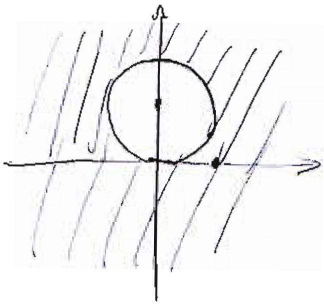
$x>0$, $y<0$ $\quad x - y = 4$

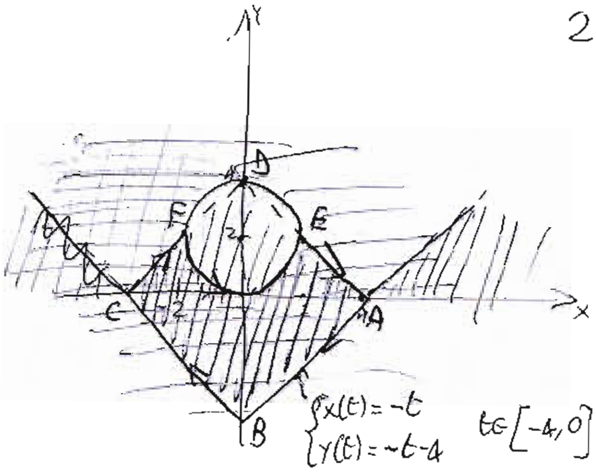
$x>0$, $y>0$ $\quad x + y = 4$

$x<0$, $y<0$ $\quad -x - y = 4$

$x<0$, $y>0$ $\quad -x + y = 4$

g) $x^2 + (y-2)^2 \geq 4$





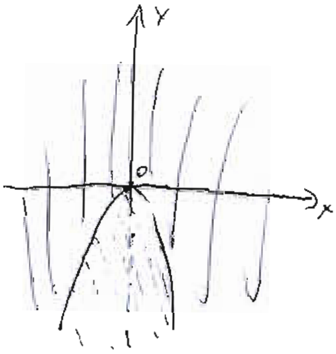
$\begin{cases} x(t) = -t \\ y(t) = -t-4 \end{cases} \quad t \in [-4, 0]$

~~$\begin{cases} |x| = |t| \\ |y| = 4 - |t| \end{cases} \quad t \in]-\infty, -2]$~~

~~$\begin{cases} x = t \\ (y-2)^2 = 4 - t^2 \end{cases} \quad t \in]-2, 2]$~~

~~$\begin{cases} |x| = |t| \\ |y| = 4 - |t| \end{cases} \quad t \in [2, +\infty[$~~

③ $y + 2x^2 > 0 \qquad 2x^2 > -y \qquad y > -2x^2$



/// esiste

$(y-x-3) \log(y+2x^2) = 0$

$y - x - 3 = 0 \qquad y = x + 3$

o

$\log(y+2x^2) = 0 \qquad y + 2x^2 = 1 \qquad y = 1 - 2x^2$

~~$f = 0 \qquad y = x+3 \cap y = -2x^2+1$~~

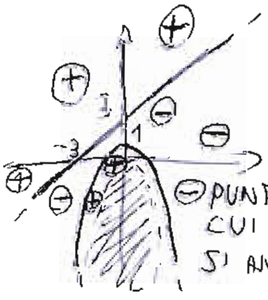
~~$f \leq 0 \qquad 2x^2+1 \leq y \leq x+3$~~

~~$f \geq 0 \qquad -2x^2 < y \leq -2x^2+1 \cup x+3 \leq y$~~

$f = 0 \quad (y-x-3) \cdot \log(y+2x^2) = 0$

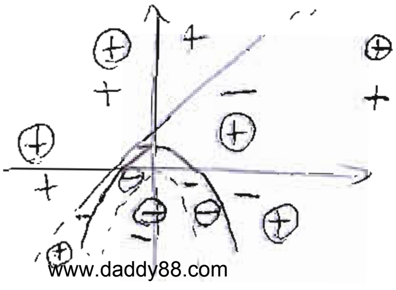
$y - x - 3 = 0 \qquad y = x + 3$

$y + 2x^2 = 1 \qquad y = -2x^2 + 1$



⊖ PUNTI CUI f SI ANNU

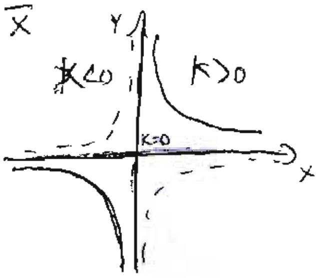
$f > 0 \quad \begin{cases} y - x - 3 > 0 \\ y + 2x^2 > 1 \end{cases} \quad$ o $\quad \begin{cases} y - x - 3 < 0 \\ y + 2x^2 \cancel{<} 1 \end{cases}$



⊕ → parabola

\+ → retta

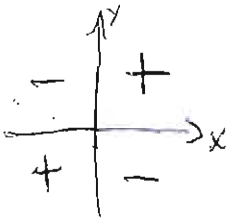
④ $xy=K \quad x\neq 0 \rightarrow y=\frac{K}{x}$



E se $k=0$?

$xy=0 \quad x=0$ o $y=0$

$xy\geq 0$



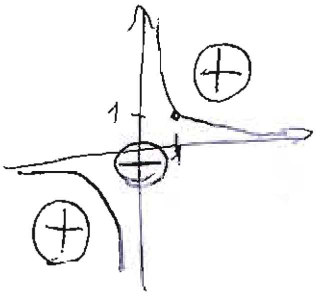
$xy=1 \quad y=\frac{1}{x}$

$xy\leq 1$

$x\cdot y\geq 1$

$\min_T f = f(-1,3) = -3$

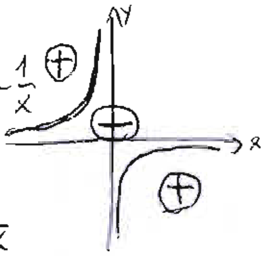
$f\leq -3 \cap T = \{(-1,3)\}$



se $x>0 \quad y\geq \frac{1}{x}$

se $x<0 \quad y\leq \frac{1}{x}$

$xy=-1 \quad x=-\frac{1}{x}$

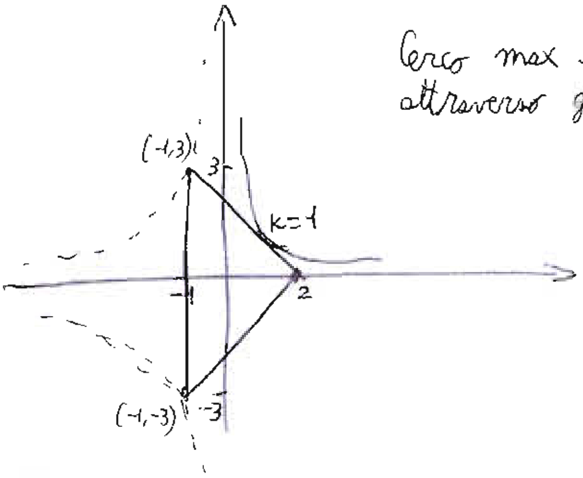


$xy\geq -1$

se $x>0 \quad y\geq -\frac{1}{x}$

se $x<0 \quad y\leq -\frac{1}{x}$

Cerco max e min di $f=xy$ attraverso gli insiemi di livello



$xy=K$ passante per $(-1,3)$

$-1\cdot 3=k \quad k=-3$

passante per $(-1,-3)$

$-1\cdot(-3)=k \quad k=3$

passante per $(2,0) \quad k=0$.

max assoluto $f(x,y)_T = +3 = f(-1,-3)$

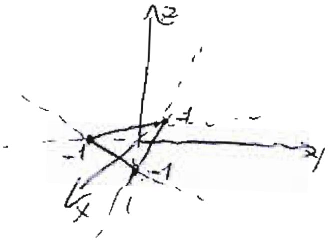
min " $f(x,y)_T = -3 = f(-1,3)$

ESERCITAZIONE

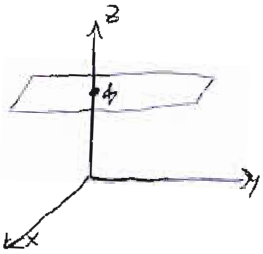
02/04/0

$Z = -x-y-1$ PIANO

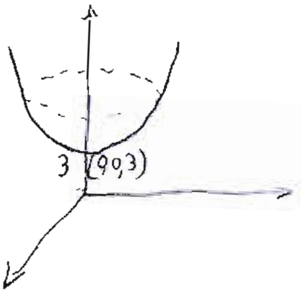
$\begin{cases} x=-1 \\ y=0 \\ z=0 \end{cases} \cap x \qquad \begin{cases} x=0 \\ y=-1 \\ z=0 \end{cases} \cap y \qquad \begin{cases} x=0 \\ y=0 \\ z=-1 \end{cases} \cap z$



$Z = 4$



$Z = x^2+y^2+3$ PARABOLOIDE CIRCOLARE CON VERTICE IN $(0,0,3)$

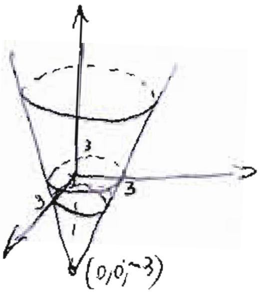


$x^2+y^2 = K-3$ circonferenze con $C(0,0)$ e raggio $\sqrt{K-3}$

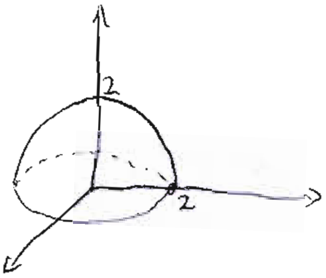
$K-3 \geq 0$ per $K<3$ non ci sono intersezioni
R aumenta all'aumentare di K.

$Z = \sqrt{x^2+y^2} - 3$ CONO CON VERTICE IN $(0,0,-3)$

$\begin{cases} z=\sqrt{x^2+y^2}-3 \\ z=0 \end{cases}$ $\cap$ piano xy $x^2+y^2=9$ circonferenza con $C(0,0$ e $R=3$



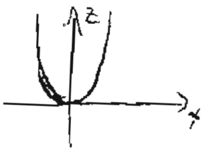
$Z = \sqrt{4-x^2-y^2}$ cioè $x^2+y^2+z^2=4 \leftarrow$ SFERA
↑
SEMISFERA



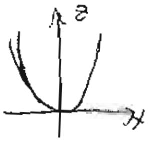
$Z = 2(x^2+y^2)$ PARABOLOIDE CIRCOLARE CON VERTICE IN $(0,0,0)$

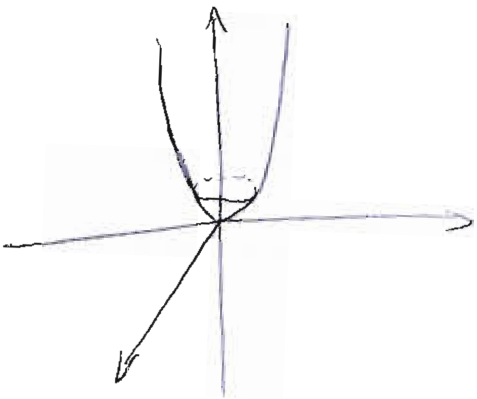
$2(x^2+y^2) = K \quad . x^2+y^2 = \frac{K}{2}$ fascio di circonferenze con centro in $(0,0)$

$\frac{K}{2} \geq 0 \quad K \geq 0$ $\begin{cases} x=0 & \text{PIANO ZY} \\ z=2y^2 \end{cases}$

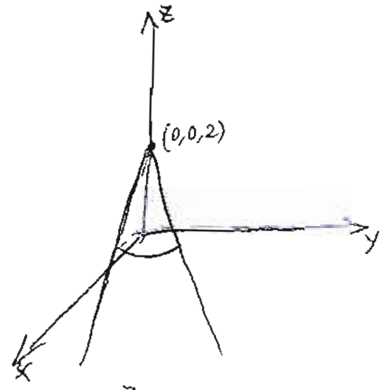


$\begin{cases} y=0 & \text{PIANO XZ} \\ z=2x^2 \end{cases}$





$Z = 2 - 3\sqrt{x^2+y^2}$ CONO DI VERTICE $(0,0,2)$ RIVOLTO VERSO IL BASSO

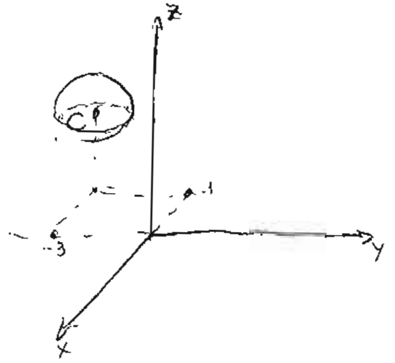


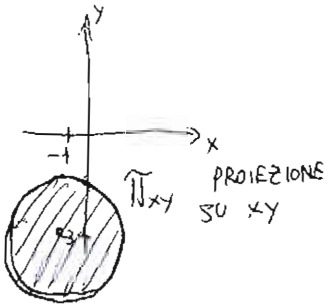
$2-3\sqrt{x^2+y^2} = K$

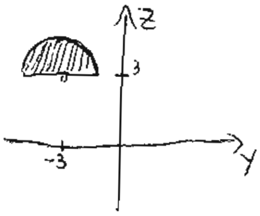
$-3\sqrt{x^2+y^2} = K-2$ $\quad \sqrt{x^2+y^2} = \frac{2}{3} - \frac{K}{3}$ $\quad x^2+y^2 = \frac{(2-K)^2}{9}$ circonferenze

$R = \frac{2-K}{3}$ se K aumenta, R diminuisce $\quad \frac{2-K}{3} \geq 0 \quad K \leq 2$

$Z = 3 + \sqrt{4-(x+1)^2-(y+3)^2}$ SEMISFERA CENTRATA IN $(-1,-3,3)$ CON RAGGIO 2

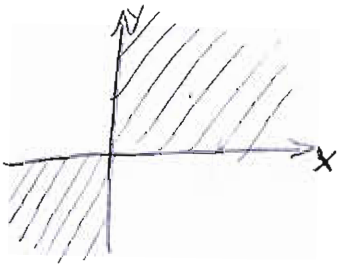






ES. N. 2.5.10 LIBRO

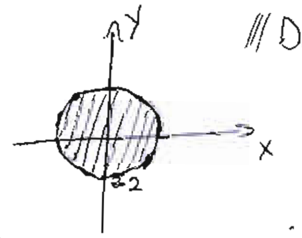
$f(xy) = \sqrt{xy}$ $\quad D = ?$ $\quad xy \geq 0$ $\quad y$



/// Dominio

$f(xy) = \frac{1}{\sqrt{4-x^2-y^2}}$ $\quad D?$ $\quad 4-x^2-y^2 > 0$

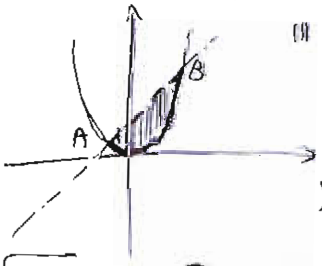
$x^2+y^2 < 4$



/// Dominio

$f(xy) = \log(1+x-y) - \sqrt{y-x^2}$ $\quad D = ?$

$\begin{cases} 1+x-y>0 \\ y-x^2 \geq 0 \end{cases}$ $\quad \begin{cases} y < 1+x \\ y \geq x^2 \end{cases}$



/// Dominio

$x_A = \frac{1-\sqrt{5}}{2}$ $\quad y_A = \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2}$

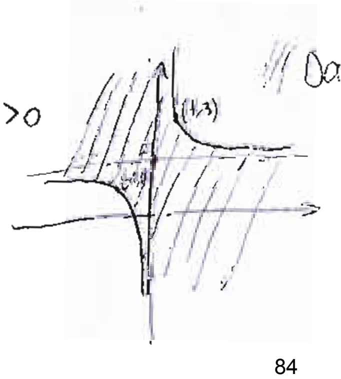
$\begin{cases} y = 1+x \\ y = x^2 \end{cases}$ $\quad x^2-x-1=0$ $\quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\quad x_B = \frac{1+\sqrt{5}}{2}$ $\quad y_B = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$

$f(x,y) = -1 + \sqrt{1-x(y-2)}$ $\quad D = ?$

$1-x(y-2) \geq 0$ $\quad xy - 2x \leq 1$ $\quad y \leq \frac{1+2x}{x}$ $\quad y \leq \frac{1}{x} + 2$ se $x>0$

$\left(-\frac{d}{c}, \frac{a}{c}\right) = (0,2)$ $\quad y \geq \frac{1}{x} + 2$ se $x<0$

$y = \frac{ax+b}{cx+d}$



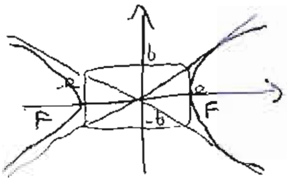
/// Dominio

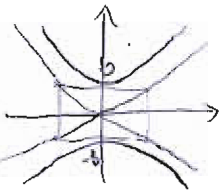
ES N° 2.5.12 02/04/08 (3

$f(x,y) = x^2 - y^2$ $\Omega = [-2,2] \times [-2,2]$ Studiare insiemi di livello

$x^2 - y^2 = K$ IPERBOLE

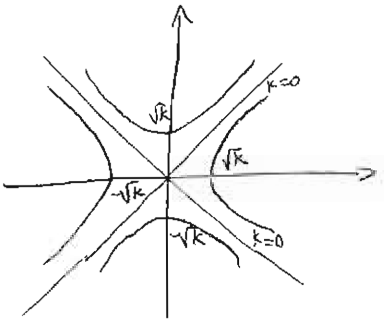
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ASSE FOCALE X $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ ASSE FOCALE Y





$x^2 - y^2 = K$ IPERBOLE EQUILATERA

Se $K = 0$ $x^2 - y^2 = 0$ $(x+y)(x-y) = 0$ $\begin{matrix} y = -x \\ y = x \end{matrix}$ ∪ BISETTRI

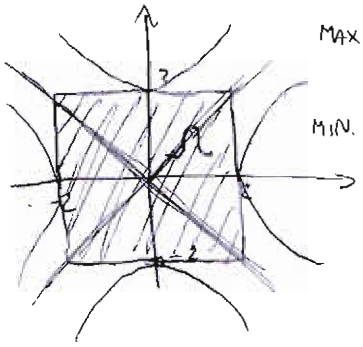


SELLA!

Se $K > 0$ è un'iperbole con fuochi in $x = \pm\sqrt{K}$

Se $K < 0$ " " " con fuochi in $y = \pm\sqrt{K}$

$x^2 - y^2 = -a^2$ $x^2 - y^2 = K$



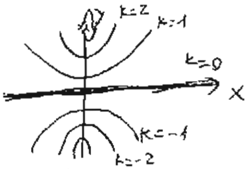
MAX $\sqrt{K} = 2$ $K = 4$ max ASS $f(x,y)_\Omega = 4 = f(2,0) = f(-2,0)$

MIN. $x^2 - y^2 = -4$ min ASS $f(x,y)_\Omega = 4 = f(0,2) = f(0,-2)$

$f(x,y) = y e^{-x^2}$ $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2y \leq 0\}$

$y e^{-x^2} = K$ $K = 0$

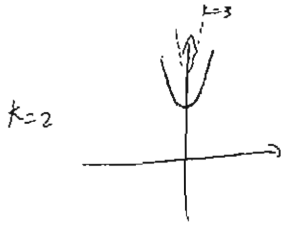
$y = K e^{x^2}$ $y = 0$



$x \to \pm\infty$ $f \to +\infty$

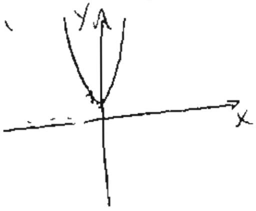
$y' = 2x \cdot e^{x^2} \geq 0$ se $x \geq 0$

$y'' = 2e^{x^2} + 4x^2 \cdot e^{x^2} = 2e^{x^2}(1+2x^2) \geq 0$ $\forall x$



$K = 1$

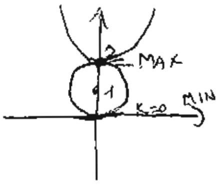
$y = e^{x^2}$



Ω

$x^2 + y^2 - 2y \leq 0$ $x^2 + y^2 - 2y + 1 - 1 \leq 0$

$x^2 + (y-1)^2 \leq 1$



min ASS $f(x,y)_\Omega = 0 = f(0,0)$

max ASS $f(x,y)_\Omega = 2 = f(0,2)$

ESERCITAZIONE 3/04/2008

1

$f(x,y) = \frac{x^2y^4}{1+x^4+y^2}$ $\quad \varphi_1(t)=(0,t) \quad \varphi_2(t)=(t,t) \quad \varphi_3(t)=(3t,4t^2) \quad \varphi_4(t)=(t,t^2) \quad \varphi_5(t)=(t^2,t)$

$\lim_{t\to\infty} f(\varphi_1(t)) = \lim_{t\to\infty} \frac{0^2t^4}{1+0^4+t^2} = 0 \qquad \lim_{t\to\infty} f(\varphi_2(t)) = \lim_{t\to\infty} \frac{t^6}{1+t^4+t^2} = \infty$

$\lim_{t\to\infty} f(\varphi_3(t)) = \lim_{t\to\infty} \frac{9t^2 \cdot 256t^8}{1+81t^4+16t^4} = \infty \qquad \lim_{t\to\infty} f(\varphi_4(t)) = \lim_{t\to\infty} \frac{t^2 \cdot t^8}{1+t^4+t^4} = \infty$

$\lim_{t\to\infty} f(\varphi_5(t)) = \lim_{t\to\infty} \frac{t^4 \cdot t^4}{1+t^8+t^2} = 1 \qquad \lim_{\|(x,y)\|\to\infty} f(x,y) \nexists$ perchè i limiti con $\varphi_1, \varphi_2, \varphi_5$ sono di

$\boxed{2}$ $\Omega =]-1,1] \times [0,1[$

1) $C(\Omega) = (]-\infty,-1] \times \mathbb{R}) \cup (]1,+\infty[\times \mathbb{R}) \cup (]-1,1] \times [1,+\infty[) \cup (]-1,1] \times]-\infty,-1[)$ FALSO

2) $\partial C(\Omega) = ([-1,1] \times \{0\}) \cup ([-1,1] \times \{1\}) \cup (\{-1\} \times]0,1[) \cup (\{1\} \times]0,1[)$ ~~FALSO~~ VERO

$\{-1,0\}, (-1,1), (1,0), (1,1)$ già inclusi nei pri due insiemi

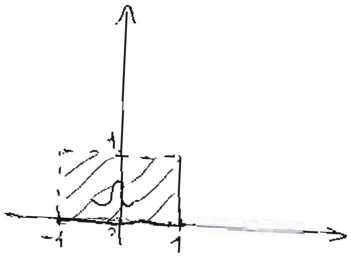
$\boxed{\partial\Omega = \partial C\Omega}$

3) Ω è chiuso FALSO

4) $\Omega \cup \partial C(\Omega)$ è chiuso VERO

5) $(1,1) \in \Omega$ FALSO

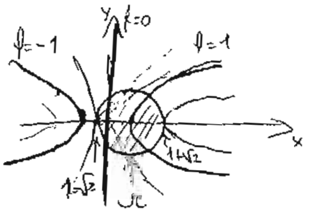
6) $(1,0)$ è punto di accumulazione per $C(\Omega)$ VERO



$\boxed{3}$ $f(x,y) = \frac{x}{1+y^2}$ Disegnare gli insiemi di livello $\{f=0\}, \{f=-1\}, \{f=1\}$ $\quad \frac{x}{1+y^2} = K$

$\Omega = \{(x,y) : x^2+y^2-2x \leq 1\}$ Determinare il massimo e il minimo di f su Ω.

$l=0 \quad \frac{x}{1+y^2} = 0 \quad x=0$



$f=1 \quad \frac{x}{1+y^2} = 1 \quad x = y^2+1$

$\Omega : (x-1)^2+y^2 \leq 2$

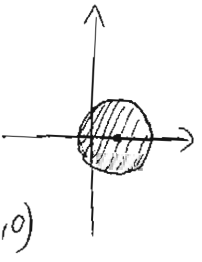
$=-1 \quad \frac{x}{1+y^2} = -1 \quad x = -1-y^2$

$x = -y^2-1$

se $K=0$ asse y

se $K>0$ parabole $x = Ky^2+K$

se $K<0$ parabole $x = Ky^2+K \quad V(K,0)$



$\Omega \cap X \quad \begin{cases} y=0 \\ x^2-2x+1=2 \end{cases} \quad x^2-2x-1=0 \quad x=\frac{\pm\sqrt{1+1}}{1}=1\pm\sqrt{2}$

$\begin{cases} x=1-\sqrt{2} \\ y=0 \end{cases}$

$K=1-\sqrt{2} \quad \frac{x}{1+y^2}=1-\sqrt{2}$ CURVA DI LIVELLO MINIMO CHE ATTRAVERSA Ω

MAX $f(x,y)_\Omega = f(1+\sqrt{2},0) = 1+\sqrt{2}$

min $f(x,y)_\Omega = f(1-\sqrt{2},0) = 1-\sqrt{2}$

Funzione continua, insieme chiuso e limitato $\Rightarrow$ esistono max e min pe il teorema di Weierstrass

---

$\boxed{4}$ Data $f(x,y)=\frac{2x-2y}{x^2+y^2+1}$ $\qquad \frac{2x-2y}{x^2+y^2+1}=K$

i) Disegnare gli insiemi $\{f=0\}, \{f=-1\}, \{f=1/2\}$

ii) Disegnare gli insiemi $\{f\leq 0\}, \{f\leq -1\}, \{f\geq 1/2\}$

$f=0 \quad \frac{2x-2y}{x^2+y^2+1}=0 \quad 2x-2y=0 \quad y=x$

$f=-1 \quad \frac{2x-2y}{x^2+y^2+1}=-1 \quad 2x-2y=-x^2-y^2-1$

$x^2+2x+1-1+y^2-2y+1\cancel{-1}\cancel{+1}=0$

$(x+1)^2+(y-1)^2=1$

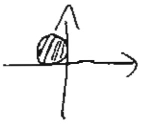
$f=\frac{1}{2} \quad \frac{2x-2y}{x^2+y^2+1}=\frac{1}{2} \quad 4x-4y=x^2+y^2+1$

$x^2-4x+4-4+y^2+4y+4-4+1=0$

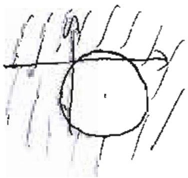
$(x-2)^2+(y+2)^2=7 \quad R=\sqrt{7}$

$f\leq 0 \quad \frac{2x-2y}{x^2+y^2+1}\leq 0 \quad 2x-2y\leq 0 \quad y\geq x$

$f\leq -1 \quad (x+1)^2+(y-1)^2\leq 1$



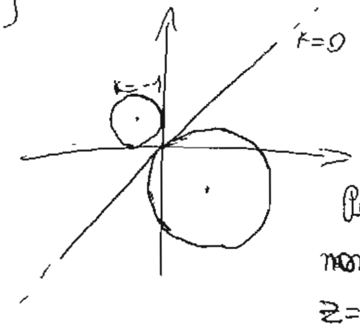
$f\geq \frac{1}{2} \ldots (x-2)^2+(y+2)^2\leq 7$



$D: x^2+y^2+1\neq 0 \quad D:\mathbb{R}$

$Kx^2+Ky^2+K-2x+2y=0 \quad$ se $K=0 \quad x=y$

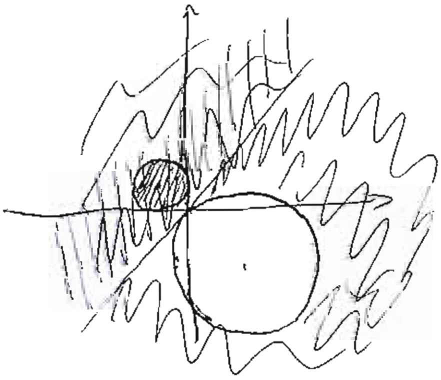
se $K\neq 0 \quad x^2+y^2-\frac{2}{K}x+\frac{2}{K}y+1=0 \quad C\left(\frac{1}{K},-\frac{1}{K}\right)$



$R=\sqrt{\frac{2}{K^2}-1}=$

$=\sqrt{\frac{2-K^2}{K^2}} \quad -\sqrt{2}\leq K\leq\sqrt{2}$

Per $K\leq-\sqrt{2}$ o $K>\sqrt{2}$, $f(x,y)$ non interseca mai il piano $z=K$



ESERCITAZIONE

09/04/0

①

Es. 2.5.22

(i) $\lim\limits_{\|(x,y)\|\to+\infty} \frac{xy^4}{x^2+y^6}$ $\quad f(0,y)=0\ \forall y \quad \lim\limits_{|y|\to+\infty} f(0,y)=0\ ;\ f(x,0)=0\ \forall x\ \lim\limits_{|x|\to+\infty} f(x,0)=0$

$$f(x,x)=\frac{x^5}{x^2+x^6}=\frac{x^{\cancel{8}3}}{\cancel{x^2}(x^4+1)}=\frac{x^3}{x^4+1} \quad \lim_{|x|\to+\infty} f(x,x)=0$$

$$f(x^2,x)=\frac{x^6}{x^4+x^6} - \lim_{x\to\infty}\frac{x^6}{x^4+x^6}=\lim_{x\to\infty}\frac{\cancel{x^6}}{\cancel{x^6}\left(\frac{1}{x^2}+1\right)}=\underline{\underline{1}}$$

$$\Rightarrow \nexists \lim_{\|(x,y)\|\to+\infty}\frac{xy^4}{x^2+y^6}$$

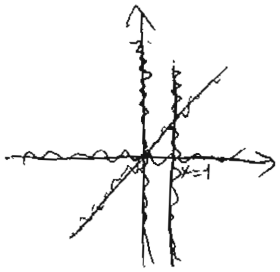
(ii) $\lim\limits_{\|(x,y)\|\to+\infty} \frac{y^2}{x^2+y^2+1}$ $\quad f(0,y)=\frac{y^2}{y^2+1}\longrightarrow 1$ per $|y|\to+\infty$ $\quad \nexists$

$f(x,0)=\frac{0}{x^2+1}\longrightarrow 0$ per $|x|\to+\infty$

(iii) $\lim\limits_{\|(x,y)\|\to+\infty} \frac{\operatorname{sen}(xy)}{|x|+1}$ $\quad f(0,y)=\frac{0}{0+1}=0\longrightarrow 0$ per $|y|\to+\infty$

$f(x,0)=0\longrightarrow 0$ per $|x|\to+\infty$

$f(x,x)=\frac{\operatorname{sen}x^2}{|x|+1}\longrightarrow 0$ per $|x|\to+\infty$



$\Rightarrow \nexists \lim$ ↑ $\quad f(1,y)=\frac{\operatorname{sen}y}{1+1}=\frac{\operatorname{sen}y}{2}\ \nexists \lim$ per $|y|\to+\infty$

(iv) $\lim\limits_{\|(x,y)\|\to+\infty} \frac{xy\cos(x^2-y^2)}{x^4+y^4}$ $\quad f(0,y)=0\ \forall y\neq 0 \quad f(x,0)=0\ \forall x\neq 0 \quad \lim\limits_{|y|\to+\infty} f(0,y)=0 \quad \lim\limits_{|x|\to+\infty} f(x,0)=0$

$f(x,x)=\frac{x^2\cancel{\cos 0}}{x^4+x^4}=\frac{1}{2x^2}\longrightarrow 0$ per $|x|\to+\infty$

Assumo che il limite valga 0

Lo dimostro!

$\begin{cases} x=\rho\cos\theta \\ y=\rho\operatorname{sen}\theta \end{cases}$

$$\leq |f(x,y)|=\left|\frac{xy\cos(x^2-y^2)}{x^4+y^4}\right|=\left|\frac{xy}{x^4+y^4}\right|\cdot\underbrace{\left|\cos(x^2-y^2)\right|}_{\leq 1}\leq\left|\frac{xy}{x^4+y^4}\right|\overset{C.P.}{=}\left|\frac{\rho^2\cos\theta\operatorname{sen}\theta}{\rho^4\cos^4\theta+\rho^4\operatorname{sen}^4\theta}\right|=$$

$$=\frac{\rho^2}{\rho^{\cancel{4}2}}\cdot\frac{\overset{\leq 1}{|\cos\theta|}\,\overset{\leq 1}{|\operatorname{sen}\theta|}}{(\cos^4\theta+\operatorname{sen}^4\theta)}\leq\frac{1}{\rho^2|\cos^4\theta+\operatorname{sen}^4\theta|}\leq\frac{2}{\rho^2}\xrightarrow[\rho\to+\infty]{}0$$

T. carabinieri

$|\cos^4\theta+\operatorname{sen}^4\theta|\geq\frac{1}{2}$

$\frac{1}{|\cos^4\theta+\operatorname{sen}^4\theta|}\leq 2$

$\hookrightarrow \cos^4\theta+\operatorname{sen}^4\theta+2\operatorname{sen}^2\theta\cos^2\theta-2\operatorname{sen}^2\theta\cos^2\theta=\underbrace{(\cos^2\theta+\operatorname{sen}^2\theta)}_{=1}-2\operatorname{sen}^2\theta\cos^2\theta=$

$=1-2\operatorname{sen}^2\theta\cos^2\theta=1-\frac{1}{2}\cdot 4\operatorname{sen}^2\theta\cos^2\theta=1-\frac{1}{2}\operatorname{sen}^2 2\theta\geq 1-\frac{1}{2}=\frac{1}{2}$

$\hookrightarrow 2\operatorname{sen}\theta\cos\theta=\operatorname{sen}2\theta$

$\operatorname{sen}^2 2\theta\leq 1$

$-\operatorname{sen}^2 2\theta\geq -1$

ES. 3.5.1.

Calcolare le derivate direzionali di f rispetto al versore $v = (\cos\theta, \operatorname{sen}\theta)$, $\theta \in [0, 2\pi]$ IN P.

(i) $f(x,y) = (x^2-y)e^{xy-2}$ $P \equiv (1,0)$

$$\frac{\partial f}{\partial x}(x,y) = 2x e^{xy-2} + (x^2-y)\cdot e^{xy-2}\cdot y = e^{xy-2}(2x + x^2y - y^2) \text{ continua}$$

$$\frac{\partial f}{\partial y}(x,y) = -1\cdot e^{xy-2} + (x^2-y)e^{xy-2}\cdot x = e^{xy-z}(x^3 - xy - 1) \text{ continua}$$

$\Rightarrow f$ differenziabile

$$\frac{\partial f}{\partial x}(1,0) = e^{-2}(2) = 2e^{-2} = \frac{2}{e^2} \qquad \frac{\partial f}{\partial y}(1,0) = e^{-2}(1-0-1) = 0$$

$$\frac{\partial f}{\partial v}(1,0) = \frac{2}{e^2}\cdot\cos\theta + 0\cdot\operatorname{sen}\theta = \frac{2\cos\theta}{e^2}$$

PAG. 97 (iii)

$$\frac{\partial f}{\partial v}(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\cdot v_1 + \frac{\partial f}{\partial y}(x_0$$

PAG. 99 TEOREMA 3.2.10

SE ESISTONO LE DERIVATE PARZ

IN UN INTORNO DI (x_0,y_0) E SONO

CONTINUE IN (x_0,y_0) ALLORA f È

DIFFERENZIABILE IN (x_0,y_0)

① calcolo le derivate parzi

② se sono continue, trovo le derivate direzionali

(ii) $f(x,y) = \underbrace{xy}\cdot\underbrace{\log(x^2+y^2)}$ $P \equiv (0,1)$

$$\frac{\partial f}{\partial x}(x,y) = y\log(x^2+y^2) + xy\,\frac{1}{x^2+y^2}\cdot 2x = y\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right] \text{ continua } \forall (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y}(x,y) = x\log(x^2+y^2) + xy\,\frac{1}{x^2+y^2}\cdot 2y = x\left[\log(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right] \text{ continua } \forall (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,1) = 1\left[0 + \frac{0}{0+1}\right] = 0 \qquad \frac{\partial f}{\partial y}(0,1) = 0 \qquad \frac{\partial f}{\partial v}(0,1) = 0$$

(iii) $f(x,y) = xy\cdot\log(x^2+y^2)$ $P \equiv \left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

$$\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) = \frac{1}{\sqrt{2e}}\left(\log\left(\frac{1}{2e} + \frac{1}{2e}\right) + \frac{2\,\frac{1}{2e}}{\frac{1}{2e}+\frac{1}{2e}}\right) = \frac{1}{\sqrt{2e}}\left(\log e^{-1} + \frac{\frac{1}{e}}{\frac{1}{e}}\right) = \frac{1}{\sqrt{2e}}(-1+1) = 0$$

$$\frac{\partial f}{\partial y}\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial v}\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right) = 0.$$

(iv) $f(x,y) = \underbrace{xy^2}\underbrace{\arctan(x^2y)}$ $P \equiv (1,1)$

$$\frac{\partial f}{\partial x}(x,y) = y^2\arctan(x^2y) + xy^2\cdot\frac{2xy}{1+x^4y^2} = y^2\left(\arctan(x^2y) + \frac{2x^2y}{1+x^4y^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = 2xy\arctan(x^2y) + xy^2\cdot\frac{x^2}{1+x^4y^2} = xy\left(2\arctan(x^2y) + \frac{x^2y}{1+x^4y^2}\right)$$

$$\frac{\partial f}{\partial x}(1,1) = 1\left(\frac{\pi}{4}+1\right) = \frac{\pi}{4}+1 \qquad \frac{\partial f}{\partial y}(1,1) = 1\left(2\cdot\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{2} + \frac{1}{2} \qquad \frac{\partial f}{\partial v}(1,1) = \left(\frac{\pi}{4}+1\right)\cos\theta + \left(\frac{\pi}{2}+\frac{1}{2}\right)\operatorname{sen}\theta$$

05/04/2008

②

EQUAZIONE PIANO TANGENTE A $z=f(x,y)$ IN $P(x_0,y_0,f(x_0,y_0))$

$$z=f(x_0,y_0)+\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

ES N. 3.5.6

i) $f(x,y)=(x^2-y)e^{xy-2}$ $\quad P\equiv(1,0)$ TROVARE z (PIANO TANGENTE)

$$\frac{\partial f}{\partial x}=2xe^{xy-2}+(x^2-y)e^{xy-2}\cdot y=e^{xy-2}(2x+x^2y-y^2)$$ funzioni

$$\frac{\partial f}{\partial y}=-1\cdot e^{xy-2}+(x^2-y)e^{xy-2}\cdot x=e^{xy-2}(x^3-xy-1)$$ continue

$$\frac{\partial f}{\partial x}(1,0)=e^{-2}(2)=\frac{2}{e^2}\qquad\frac{\partial f}{\partial y}(1,0)=e^{-2}(1-0-1)=0$$

$$f(1,0)=(1-0)e^{1-0-2}=e^{-2}=\frac{1}{e^2}\qquad z=\frac{1}{e^2}+\frac{2}{e^2}(x-1)=\frac{2}{e^2}x-\frac{1}{e^2}$$ PIANO TANGENTE

(ii) $f(x,y)=\log(x+y^2+1)$ $\quad P\equiv(0,0)$

$$\frac{\partial f}{\partial x}(x,y)=\frac{1}{x+y^2+1}\qquad\frac{\partial f}{\partial y}(x,y)=\frac{2y}{x+y^2+1}\qquad\frac{\partial f}{\partial x}(0,0)=1\qquad\frac{\partial f}{\partial y}(0,0)=0$$

$$f(0,0)=0\qquad z=0+1(x-0)+0(y-0)\Rightarrow\boxed{z=x}$$ PIANO TANGENTE

(iii) $f(x,y)=\underbrace{xy}\,\underbrace{\log(x^2+y^2)}$ $\quad P\equiv(0,1)$

$$\frac{\partial f}{\partial x}=y\log(x^2+y^2)+xy\frac{2x}{x^2+y^2}\qquad\frac{\partial f}{\partial x}(0,1)=0\qquad f(0,1)=0\qquad\boxed{z=0.}$$

$$\frac{\partial f}{\partial y}=x\log(x^2+y^2)+xy\frac{2y}{x^2+y^2}\qquad\frac{\partial f}{\partial y}(0,1)=0$$

(iV) $f(x,y)=xy\log(x^2+y^2)$ $\quad P=\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

$$f\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)=\frac{1}{2e}\log\left(\frac{1}{2e}+\frac{1}{2e}\right)=\frac{1}{2e}\log e^{-1}=-$$

$$\frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)=0\qquad\frac{\partial f}{\partial y}\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)=0$$

$$\boxed{z=-\frac{1}{2e}}$$ PIANO TANGENTE IN $\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}},-\frac{1}{2e}\right)$

(V) $f(x,y)=x\sqrt{y^2-xy}$ $\quad P\equiv(3,4)$

$$\frac{\partial f}{\partial x}(x,y)=\sqrt{y^2-xy}+x\frac{1}{2\sqrt{y^2-xy}}\cdot(-y)=\frac{2(y^2-xy)-xy}{2\sqrt{y^2-xy}}=\frac{2y^2-3xy}{2\sqrt{y^2-xy}}\qquad\frac{\partial f}{\partial x}(3,4)=\frac{32-36}{2\sqrt{16-12}}=\frac{-4}{2\cdot 2}=-1$$

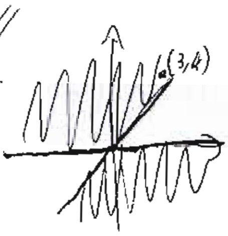
$$\frac{\partial f}{\partial y}(x,y)=x\cdot\frac{2y-x}{2\sqrt{y^2-xy}}=\frac{2xy-x^2}{2\sqrt{y^2-xy}}\qquad\frac{\partial f}{\partial y}(3,4)=\frac{24-9}{2\sqrt{16-12}}=\frac{15}{4}\qquad f(3,4)=3\sqrt{16-12}=6$$

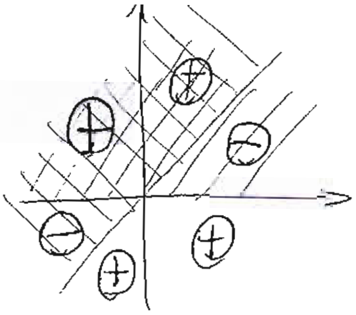
$$z=6-1\cdot(x-3)+\frac{15}{4}(y-4)=6-x+3+\frac{15}{4}y-15$$

$$z=-x+\frac{15}{4}y-6$$

DOM: $y^2 - xy \geq 0$ $\quad y(y-x) \geq 0$ $\quad y \geq 0$ //// $\quad y \geq x$ \\\\

Mentre per $y=0$ o $y=x$, la f esiste ma non esistono le derivate parziali





TROVARE LA DERIVATA DIREZIONALE SECONDO IL VERSORE $V(V_1, V_2)$ DELLA FUNZIONE $f(x,y) = 3xy^2 + y^2 - 3x - 6y + 7$ A) CON IL TEOREMA DEL DIFFERENZIALE TOTALE + FORMULA E B) COME LIMITE (P.94) IN $P \equiv (0,1)$ $\quad f(0,1) = 2$

A) $\frac{\partial f}{\partial x}(x,y) = 3y^2 - 3$ $\quad \frac{\partial f}{\partial y}(x,y) = 6xy + 2y - 6$ $\quad \frac{\partial f}{\partial x}(0,1) = 0$ $\quad \frac{\partial f}{\partial y}(0,1) = -4$ $\quad \frac{\partial f}{\partial V}(0,1) = -4V_2$

B) $$\lim_{t \to 0} \frac{f(tv_1, 1+tv_2) - f(0,1)}{t} = \lim_{t \to 0} \frac{3tv_1(1+tv_2)^2 + (1+tv_2)^2 - 3tv_1 - 6(1+tv_2) + 7 - 2}{t} =$$

$$= \lim_{t \to 0} \frac{3tv_1(1+t^2v_2^2+2tv_2) \cancel{+1} + t^2v_2^2 + 2tv_2 - 3tv_1 \cancel{-6} - 6tv_2 \cancel{+5}}{t} =$$

$$= \lim_{t \to 0} \frac{\cancel{3tv_1} + 3t^{\overset{2}{\cancel{3}}}v_1v_2^2 + 6t^{\cancel{2}}v_1v_2 + t^{\cancel{2}}v_2^2 + 2\cancel{t}v_2 \cancel{-3tv_1} - 6\cancel{t}v_2}{\cancel{t}} = \lim_{t \to 0} (3t^2 v_1v_2^2 + 6tv_1v_2 + tv_2^2 + 2v_2 - 6v_2) =$$

$= -4V_2$ ▨

ESERCITAZIONE

10/04/200

① Calcolare il gradiente ∇f delle seguenti funzioni, specificando dove esiste.

- $F_1(x,y) = 3x^2y + 3y^3 + \frac{3x}{1+y^2}$ $\quad \frac{\partial F}{\partial x}(x,y) = 6xy + \frac{3}{1+y^2}$ $\quad \frac{\partial F}{\partial y} = 3x^2 + 9y^2 + \frac{-2y \cdot 3x}{(1+y^2)^2}$

$$\boxed{\nabla F_1(x,y) = \left(6xy + \frac{3}{1+y^2} ; 3x^2 + 9y^2 - \frac{6xy}{(1+y^2)^2}\right)}$$

$\nabla F_1(x,y)$ esiste $\forall (x,y) \in \mathbb{R}^2$

- $F_2(x,y) = xe^{y^2} + y\log(xy)$ $\quad$ D: $xy > 0$ $\quad \begin{matrix} x>0 \\ y>0 \end{matrix}$ o $\begin{matrix} x<0 \\ y<0 \end{matrix}$

$\frac{\partial F}{\partial x} = e^{y^2} + y\frac{1}{xy} \cdot x$ $\quad \frac{\partial f}{\partial y} = xe^{y^2} \cdot 2y + \log(xy) + y \cdot \frac{1}{xy} \cdot x$

$$\boxed{\nabla F_2(x,y) = \left(e^{y^2} + \frac{y}{x} ; 2xye^{y^2} + \log(xy) + 1\right)}$$

$\nabla F_2(x,y)$ esiste $\forall (x,y) \in \mathbb{R}^2 : x>0, y>0$ o $x<0, y<0$.

- $f_3(x,y) = e^{\operatorname{sen} y} + y^2 \log x$ $\quad$ D: $x > 0$ $\quad$ esiste $\forall (x,y) \in \mathbb{R}^2 : x > 0$

$\frac{\partial f}{\partial x} = \frac{y^2}{x}$ $\quad \frac{\partial f}{\partial y} = e^{\operatorname{sen} y} \cdot \cos y + 2y \log x$ $\quad \boxed{\nabla f_2(x,y) = \left(\frac{y^2}{x} ; e^{\operatorname{sen} y} \cdot \cos y + 2y \log x\right)}$

② $f(x,y) = 2y + xy - x^2 - y^2$

(I) calcolare $\nabla f(x,y)$

(II) calcolare $\frac{\partial f}{\partial v}(-1,3)$ quando $v_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ e $v_2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(III) calcolare l'equazione del piano tangente in $(1, -1, f(1,-1))$

(I) $\frac{\partial f}{\partial x} = y - 2x$ $\quad \frac{\partial f}{\partial y} = 2 + x - 2y$ $\quad \boxed{\nabla f(x,y) = (y - 2x ; 2 + x - 2y)}$

(II) $\frac{\partial f}{\partial v_1}(x,y) = \frac{\sqrt{2}}{2}(y - 2x) + \frac{\sqrt{2}}{2}(2 + x - 2y)$ $\quad \frac{\partial f}{\partial v_1}(-1,3) = \frac{5\sqrt{2}}{2} + \frac{-5\sqrt{2}}{2} = \boxed{0}$

$$\frac{\partial f}{\partial v_2}(-1,3) = -\frac{\sqrt{3}}{2}(3+2) + \frac{1}{2}(2-1-6) = -\frac{5\sqrt{3}}{2} - \frac{5}{2} = \frac{-5\sqrt{3} - 5}{2} = \boxed{-\frac{5}{2}(\sqrt{3}+1)}$$

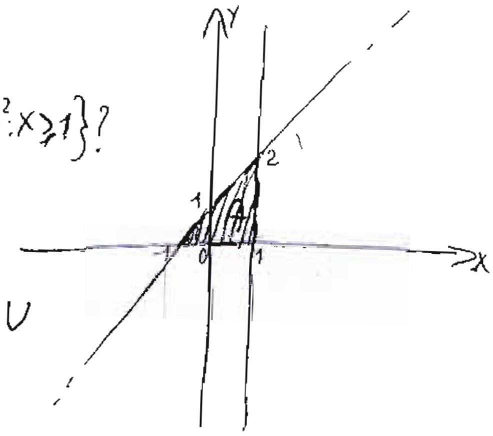
(III) $f(1,-1) = -2 - 1 - 1 - 1 = -5$

$z = -5 + (-1-2)(x-1) + (2+1+2)(y+1)$ $\quad z = \cancel{-5} - 3x + 3 + 5y \cancel{+5}$

$$\boxed{z = -3x + 5y + 3}$$

④ $A_1(x,y)=\{(x,y)\in\mathbb{R}^2 : y-x<1\}$ $A_2(x,y)=\{(x,y)\in\mathbb{R}^2 : y>0\}$ $A_3(x,y)=\{(x,y)\in\mathbb{R}^2 : x-1<0\}$

(I) Rappresentare graficamente $A = A_1 \cap A_2 A A_3$



II) $CA=\{(x,y)\in\mathbb{R}^2 : y-x\geq 1\}\cup\{(x,y)\in\mathbb{R}^2 : y\leq 0\}\cup\{(x,y)\in\mathbb{R}^2 : x\geq 1\}$?

<u>VERO</u>

III) $\partial CA=\{(1,t)\in\mathbb{R}^2 : t\in[0,2]\}\cup\{(t,0)\in\mathbb{R}^2 : t\in[-1,1]\}\cup$
$\{(t,t+1)\in\mathbb{R}^2 : t\in[0,1]\}$? FALSO

$[-1,1]$

IV) A è chiuso? FALSO

) A è aperto? VERO

V) $A\cup\delta CA$ è chiuso? VERO

③

$f(x,y)=x^3+y^3+3xy-4$

) calcolate, se esiste, $\lim_{\|(x,y)\|\to\infty} f$; nel caso non esista, motivare perché.

$f(0,t)=t^3-4$ $\lim_{t\to\pm\infty} f(0,t)=\pm\infty$ $f(t,0)=t^3-4$ $\lim_{t\to\pm\infty} f(t,0)=\pm\infty$ $\boxed{\lim_{\|(x,y)\|\to\infty} f(x,y) \nexists}$

$F(t,t)=t^3+t^3+3t^2-4=$ $\lim_{t\to\pm\infty} f(t,t)=\pm\infty$ $f(1,t)=1+t^3+3t-4$ $\lim_{t\to\infty} f(1,t)=\infty$
$=2t^3+3t^2-4$

Il limite non esiste in quanto i limiti per $t\to+\infty$ e $t\to-\infty$ sono diversi

) Calcolare ∇f

$\frac{\partial f}{\partial x}=3x^2+3y$ $\frac{\partial f}{\partial y}=3y^2+3x$ $\boxed{\nabla f=(3x^2+3y, 3y^2+3x)}$

) calcolare la derivata direzionale di f nel punto $(-1,1)$ rispetto alla direzione $v\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

$\frac{\partial f}{\partial v}(-1,1)=(3+3)\cdot\left(-\frac{1}{2}\right)+(3-3)\cdot\frac{\sqrt{3}}{2}=\boxed{-3}$

calcolare l'equazione del piano tangente a f nel punto $(-1,1,f(-1,1))$. $f(-1,1)=-1+1-3-4=-7$

$z=-7+(3+3)\cdot(x+1)+(0)\cancel{(y-1)}$ $z=-7+6x+6$ $\boxed{z=6x-1}$

⑤ Calcolare $\frac{\partial f}{\partial v}(-1,3)$ con $v=\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ come limite e confrontarlo con il risultato ottenuto in precedenza.

$f(x,y) = 2y + xy - x^2 - y^2$

$f(-1,3) = 6 - 3 - 1 - 9 = -7$

$$\lim_{t\to 0} \frac{f(x_0+tv_1, y_0+tv_2) - f(x_0,y_0)}{t} = \frac{\partial f}{\partial v}(x_0,y_0)$$

$$\lim_{t\to 0} \frac{2\left(3+\frac{1}{2}t\right) + \left(-1-\frac{\sqrt{3}}{2}t\right)\left(3+\frac{1}{2}t\right) - \left(-1-\frac{\sqrt{3}}{2}t\right)^2 - \left(3+\frac{1}{2}t\right)^2 + 7}{t} =$$

$$= \lim_{t\to 0} \frac{\cancel{6} + t \cancel{-3} - \frac{1}{2}t - \frac{3\sqrt{3}}{2}t - \frac{\sqrt{3}}{4}t^2 \cancel{-1} - \frac{3}{4}t^2 - \sqrt{3}t \cancel{-9} - \frac{1}{4}t^2 - 3t \cancel{+7}}{t} =$$

$$\lim_{t\to 0} \frac{\cancel{t}\left(1 - \frac{1}{2} - \frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{4}t - \frac{3}{4}t - \sqrt{3} - \frac{1}{4}t - 3\right)}{\cancel{t}} = 1 - \frac{1}{2} - \frac{3\sqrt{3}}{2} - \sqrt{3} - 3 = \frac{2 - 1 - 3\sqrt{3} - 2\sqrt{3} - 6}{2} = \frac{-5-5\sqrt{3}}{2} = \boxed{= -\frac{5}{2}\left(1+\sqrt{3}\right)}$$

RISULTATO
ATTESO

$\boxed{3.5.28}$

$f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$ $(x,y)\neq(0,0)$ $f(0,0)=0$ $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$

$$\frac{\partial f}{\partial x} = \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} = \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$\frac{\partial f^2}{\partial x \partial y} = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2+y^2)^2 - (x^4y + 4x^2y^3 - y^5)(4y^3 + 4x^2y)}{(x^2+y^2)^4} = \frac{x^8 + x^4y^4 + 2x^6y^2 + 12x^6y^2 + 12x^2y^6 + 24x^4y^4 - 5x^4y^4 - 5y^8 - 10x^2y^6 + -4x^4y^4 - 4x^6y^2 - 16x^2y^6 - 16x^4y^4 + 4y^8 + 4x^2y^6}{(x^2+y^2)^4} =$$

$$= \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2+y^2)^4}$$

$$\frac{\partial f}{\partial y} = \frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2} = \frac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{(x^2+y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$\frac{\partial f^2}{\partial y \partial x} = \frac{(5x^4 - 12x^2y^2 - y^4)(x^2+y^2)^2 - (4x^3 + 4xy^2)(x^5 - 4x^3y^2 - xy^4)}{(x^2+y^2)^4} = \frac{5x^8 + \cancel{5x^4y^4} + 10x^6y^2 - 12x^6y^2 - 12x^2y^6 - \cancel{24x^4y^4} - \cancel{x^4y^4} - y^8 - 2x^2y^6 - 4x^8 + 16x^6y^2 + \cancel{4x^4y^4} - 4x^6y^2 + \cancel{16x^4y^4} + 4x^2y^6}{(x^2+y^2)^4} =$$

$$= \frac{x^8 + 10x^6y^2 - 10x^2y^6 - y^8}{(x^2+y^2)^4}$$

ESERCITAZIONE

17/04/08

①

PIANO TANGENTE

$$z - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

esiste se $\nexists \frac{\partial f}{\partial y}$ e $\frac{\partial f}{\partial x}$

RETTA TANGENTE

$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = 0$$

$z = \sqrt{x^2+y^2}$ CONO CON $V(0,0)$ piano tg in $(0,0)$

$f(0,0)=0$ $\quad \frac{\partial f}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$ $\quad \frac{\partial f}{\partial x}(0,0) = \frac{0}{0} \nexists$ $\quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}(0,0) = \frac{0}{0} \nexists$ $\quad$ $(x,y)\neq(0,0)$ × le derivate

$\lim\limits_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ PROVO VARIE RESTRIZIONI $\quad \underline{x=0}$ $\lim\limits_{y\to 0^\pm} \frac{0}{\sqrt{0+y^2}} = 0$ $\quad y=0$ $\lim\limits_{x\to 0^\pm} \frac{x}{\sqrt{x^2+0}} = \lim\limits_{x\to 0^\pm} \frac{x}{|x|} = \pm 1$

$\Rightarrow \nexists \lim \Rightarrow \nexists \frac{\partial f}{\partial x} \Rightarrow \nexists$ piano tangente

$f(x,y) = 3xy^2 + y^2 - 3x - 6y + 7$ $\quad P(0,1)$ trovare piano tangente in P e retta tangente all'insieme di livello, cioè $\begin{cases}\text{PIANO}\\ z=f(0,1)\end{cases}$

$f(0,1) = 1 - 6 + 7 = 2$

$\frac{\partial f}{\partial x} = 3y^2 - 3$ $\quad \frac{\partial f}{\partial y} = 6xy + 2y - 6$ $\quad \frac{\partial f}{\partial x}(0,1) = 0$ $\quad \frac{\partial f}{\partial y}(0,1) = -4$

$z = 2 + 0(x-0) - 4(y-1)$ $\quad z = 2 - 4y + 4$ $\quad z = -4y + 6$ $\quad$ PIANO TANGENTE

RETTA: $-4y + 4 = 0$ $\quad y = 1$ $\quad \begin{cases} x = t \\ y = 1 \end{cases} t \in \mathbb{R}$ PARAMETRIZZAZIONE

- $f(x,y) = \frac{y-x}{2+x^2+y^2}$ $\quad P(0,1)$ $\quad f(0,1) = \frac{1}{3}$ $\quad \frac{\partial f}{\partial x} = \frac{-(2+x^2+y^2)-(y-x)(2x)}{(2+x^2+y^2)^2} = \frac{-2-x^2-y^2-2xy+2x}{(2+x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = \frac{2+x^2+y^2-(y-x)(2y)}{(2+x^2+y^2)} = \frac{x^2-y^2+2xy+2}{(2+x^2+y^2)^2}$ $\quad = \frac{x^2-y^2-2xy-2}{(2+x^2+y^2)^2}$

$\frac{\partial f}{\partial x}(0,1) = \frac{-3}{9} = -\frac{1}{3}$ $\quad \frac{\partial f}{\partial y}(0,1) = \frac{1}{9}$ $\quad z = \frac{1}{3} - \frac{1}{3}(x-0) + \frac{1}{9}(y-1)$ $\quad z = -\frac{1}{3}x + \frac{1}{9}y + \frac{2}{\ }$

RETTA TANGENTE: $-\frac{1}{3}x + \frac{1}{9}(y-1) = 0$ $\quad -3x + y - 1 = 0$ $\quad y = 3x+1$ $\quad \begin{cases} x = t \\ y = 3t+1 \end{cases} t \in \mathbb{R}$

- $f(x,y) = xye^{y-x}$ $\quad P(1,1)$ $\quad f(1,1) = 1$ $\quad \frac{\partial f}{\partial x} = y\left(e^{y-x} + xe^{y-x}\cdot(-1)\right) = y\left(e^{y-x} - xe^{y-x}\right) =$

$= ye^{y-x}(1-x)$ $\quad \frac{\partial f}{\partial x}(1,1) = 0$ $\quad \frac{\partial f}{\partial y} = x\left(e^{y-x} + ye^{y-x}\right) = xe^{y-x}(1+y)$ $\quad \frac{\partial f}{\partial y}(1,1) = 2$

$z = 1 + \underbrace{0(x-1) + 2(y-1)}_{\hookrightarrow = 0 \Rightarrow \text{Retta tangente}}$ $\quad z = 2y - 1 \quad$ RETTA TANGENTE: $2y - 2 = 0 \quad y = 1$

PUNTI STAZIONARI

$\nabla f = 0$ Punti in cui il piano tangente è parallelo al piano xy.

3.5.8

i) Trovare punti stazionari, estremo superiore e inferiore

$f(x,y) = x^2 - 2x + 3y^2 - 12y + 12 \qquad \frac{\partial f}{\partial x} = 2x - 2 \qquad \frac{\partial f}{\partial y} = 6y - 12$ $\quad$ 2 equazioni in 2 incognite

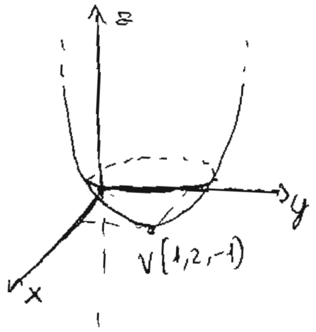
$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \quad \begin{cases} 2x - 2 = 0 \\ 6y - 12 = 0 \end{cases} \quad \begin{cases} x = 1 \ \forall y \in \mathbb{R} \\ y = 2 \ \forall x \in \mathbb{R} \end{cases}$ $P(1,2)$ è un punto stazionario

Cerco inf e sup

$f(x,y) = x^2 - 2x + 1 - 1 + 3(y^2 - 4y + 4 - 4) + 12 = (x-1)^2 - 1 \cancel{-12} + 3(y-2)^2 \cancel{+12} = (x-1)^2 + 3(y-2)^2 - 1$

PARABOLOIDE ELLITTICO

$V(1,2,-1)$



$\inf_{\mathbb{R}} f(x,y) = -1 = f(1,2) = \min f$

$\sup_{\mathbb{R}} f(x,y) = +\infty$ dimostro $\overset{x=0}{\lim_{y \to \pm\infty}} 3y^2 - 12y + 12 = +\infty$ è sufficiente

ii) $f(x,y) = (x-1)^2 + xy^2 - 1 \qquad \frac{\partial f}{\partial x} = 2(x-1) + y^2 = 2x - 2 + y^2 \quad \frac{\partial f}{\partial y} = 2xy$

$\begin{cases} 2x - 2 + y^2 = 0 \\ 2xy = 0 \end{cases}$ GRADO = 4
↳ PRODOTTO GRADI
↳ MAX 4 SOLUZIONI

$\begin{cases} x = 0 \\ y = \pm\sqrt{2} \end{cases}$ o $\begin{cases} y = 0 \\ x = 1 \end{cases}$

$A(0, \sqrt{2})$ sono punti
$B(0, -\sqrt{2})$ stazionari
$C(1,0)$

$\underline{x = 1}$

$\lim_{y \to \pm\infty} (y^2 - 1) = +\infty \qquad \sup_{\mathbb{R}^2} f(x,y) = +\infty \qquad \overset{x=-1}{\lim_{y \to \pm\infty}} 4 - y^2 - 1 = -\infty \qquad \inf_{\mathbb{R}^2} f(x,y) = -\infty$

iii) $f(x,y) = x + xy^3 \qquad \frac{\partial f}{\partial x} = 1 + y^3 \qquad \frac{\partial f}{\partial y} = 3xy^2 \qquad \begin{cases} 1 + y^3 = 0 \\ 3xy^2 = 0 \end{cases}$

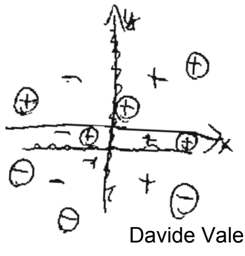
$\begin{cases} x = 0 \\ y = -1 \end{cases}$ o $\begin{cases} y = 0 \\ 1 = 0 \end{cases}$ IMP. $\quad A(0,-1)$

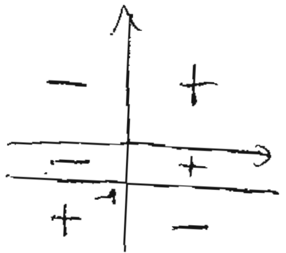
$y = 0 \quad \lim_{x \to \pm\infty} x = \pm\infty \qquad \inf_{\mathbb{R}^2} f(x,y) = -\infty$

$\sup_{\mathbb{R}^2} f(x,y) = +\infty$

, CON LO STUDIO DEL SEGNO

$(1 + y^3) \geq 0 \quad \begin{cases} x \geq 0 \\ 1 + y^3 \geq 0 \end{cases} \quad \begin{cases} x \geq 0 \\ y \geq -1 \end{cases}$





in O (zero)

in x=-1, in su va a +∞ in giù a -∞.

iv) $f(x,y)=(4x^2-2xy+y^2)e^{-2x-y}$ 17/04/08 ②

$$\frac{\partial f}{\partial x}=(8x-2y)e^{-2x-y}+(4x^2-2xy+y^2)\cdot e^{-2x-y}\cdot(-2)=e^{-2x-y}(8x-2y-8x^2+4xy-2y^2)$$

$$\frac{\partial f}{\partial y}=(-2x+2y)e^{-2x-y}+(4x^2-2xy+y^2)e^{-2x-y}(-1)=e^{-2x-y}(-2x+2y-4x^2+2xy-y^2)$$

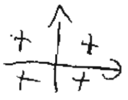
$\begin{cases}8x-2y-8x^2+4xy-2y^2=0\\-2x+2y-4x^2+2xy-y^2=0\end{cases}$ perchè $e^{-2x-y}>0$ $\forall(x,y)\in\mathbb{R}^2$ $\begin{cases}4x-y-4x^2+2xy-y^2=0\\-2x+2y-4x^2+2xy-y^2=0\end{cases}$ $\ominus$

$6x-3y=0$

$\begin{cases}y=2x\\4x-2x-\cancel{4x^2}+\cancel{4x^2}-4x^2=0\end{cases}$ $\begin{cases}y=2x\\-2x^2+x=0\end{cases}$ $\begin{cases}y=2x\\x(-2x+1)=0\end{cases}$ $\begin{cases}x=0\\y=0\end{cases}$ o $\begin{cases}x=\frac{1}{2}\\y=1\end{cases}$

$A=(0,0)$ $B=\left(\frac{1}{2},1\right)$

$(4x^2-2xy+y^2)\cdot e^{-2x-y}$ studio il segno $4x^2-2xy+y^2>0$ $x_{1,2}=\frac{2y\pm\sqrt{\overset{<0}{4y^2-16y^2}}}{8}\Rightarrow\forall x$

$e^{-2x-y}>0\ \forall x$ $\Rightarrow f(x,y)\geq0\ \forall(x,y)\in\mathbb{R}^2$  non mi serve

$\underline{x=0}$

$\lim_{x\to+\infty}y^2(e^{-y})=\lim_{y\to+\infty}\frac{y^2}{e^y}\overset{H}{=}0$ $\inf f_{\mathbb{R}^2}(x,y)$

$\lim_{y\to-\infty}y^2(e^{-y})=+\infty$ $\sup f_{\mathbb{R}^2}(x,y)$

3.5.9

iii) $f(x,y)=x(x+y)e^{y-x}$ $\frac{\partial f}{\partial x}=(2x+y)e^{y-x}+(x^2+yx)e^{y-x}\cdot(-1)=$

$=(x^2+yx)e^{y-x}$ $=e^{y-x}(2x+y-x^2-xy)$

$$\frac{\partial f}{\partial y}=xe^{y-x}+(x^2+xy)e^{y-x}=e^{y-x}(x+x^2+xy)$$

$\begin{cases}\cancel{e^{y-x}}(2x+y-x^2-xy)=0\\\cancel{e^{y-x}}(x^2+x+xy)=0\end{cases}$ $\begin{cases}-x^2+2x-xy+y=0\\x^2+x+xy=0\end{cases}$ $\begin{cases}y=-3x\\x^2+x-3x^2=0\end{cases}$ $\begin{cases}y=-3x\\x(-2x+1)=0\end{cases}$

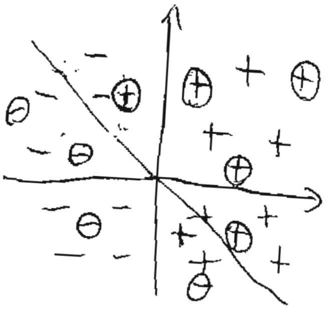
$3x+y=0$

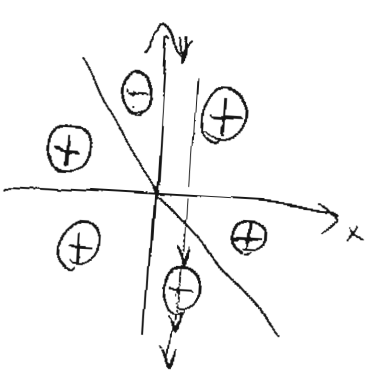
$\begin{cases}x=0\\y=0\end{cases}$ o $\begin{cases}x=\frac{1}{2}\\y=-\frac{3}{2}\end{cases}$ $A(0,0)$ $B\left(\frac{1}{2},-\frac{3}{2}\right)$ punti stazionari

studio il segno

$x(x+y)e^{y-x}\geq0$ $e^{y-x}\geq0\ \forall(x,y)\in\mathbb{R}^2$

$\begin{cases}x\geq0\\x+y\geq0\end{cases}$ $\begin{cases}x\geq0\\y\geq-x\end{cases}$





$y = x$

$\lim_{x \to \pm\infty} x(2x)e^0 = +\infty$

$\sup_{\mathbb{R}^2} f(x,y) = +\infty$

$x = 1$

$\lim_{y \to -\infty} 1 \cdot (1+y)e^{y-1} = -\infty \cdot 0$ F.I.

$\lim_{y \to -\infty} \frac{1+y}{e^{1-y}} \overset{H}{=} \lim_{y \to -\infty} \frac{1}{e^{1-y} \cdot (-1)} = 0$ me non me lo asp

$\Rightarrow$ cerco altra strada

$y = x^2$

$\lim_{x \to -\infty} x(x+x^2)e^{x^2-x} = -\infty \cdot \infty = -\infty$

ESERCITAZIONE

23/04/2008

3.5.8 (b)

①

i) $f(x,y) = x^2 - 2x + 3y^2 - 12y + 12$ $\quad \frac{\partial f}{\partial x} = 2x - 2$ $\quad \frac{\partial f}{\partial y} = 6y - 12$ $\quad A(1,2)$

$$H(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \quad \det H(x,y) = 12 > 0 \ \forall (x,y) \quad \frac{\partial^2 f}{\partial x^2} > 0$$

$\Rightarrow (1,2)$ è punto di minimo $\quad \min_{\mathbb{R}^2} f(x,y) = f(1,2) = -1$

ii) $f(x,y) = (x-1)^2 + xy^2 - 1$ $\quad \frac{\partial f}{\partial x} = 2(x-1) + y^2$ $\quad \frac{\partial f}{\partial y} = 2xy$

$A(0,\sqrt{2})$
$B(0,-\sqrt{2})$
$C(1,0)$

$$H(x,y) = \begin{bmatrix} 2 & 2y \\ 2y & 2x \end{bmatrix} \quad \det H(x,y) = 4x - 4y^2$$

$\det H(0,\sqrt{2}) = -8 < 0 \Rightarrow$ A è punto di sella

et $H(0,-\sqrt{2}) = -8 < 0 \Rightarrow$ B è punto di sella

t $H(1,0) = 4 > 0$ $\quad \frac{\partial^2 f}{\partial x^2}(1,0) = 2 > 0 \Rightarrow$ C è punto di minimo $\quad \min_{\mathbb{R}^2} f(x,y) = f(1,0) = -$

i) $f(x,y) = x + xy^3$ $\quad \frac{\partial f}{\partial x} = 1 + y^3$ $\quad \frac{\partial f}{\partial y} = 3xy^2$ $\quad A(0,-1)$

$(x,y)\begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix} \quad \det H(x,y) = -9y^4 \quad \det H(0,-1) = -9 < 0$ A è una SELLA

3.5.9

iii) $f(x,y) = x(x+y)e^{y-x}$ $\quad \dfrac{\partial f}{\partial x} = (x+y)e^{y-x} + xe^{y-x} - x(x+y)e^{y-x} = e^{y-x}(x+y+x-x^2-xy) =$

$\dfrac{\partial f}{\partial y} = xe^{y-x} + x(x+y)e^{y-x} = e^{y-x}(x+x^2+xy) = xe^{y-x}(x+y+1)$ $\quad = e^{y-x}(2x+y-x^2-xy)$

$A(0,0)$

$B(\frac{1}{2}, -\frac{3}{2})$

$\dfrac{\partial^2 f}{\partial x^2} = -e^{y-x}(2x+y-x^2-xy) + e^{y-x}(2-2x-y)$

~~MA~~ $= e^{y-x}(-2x-y+x^2+xy+2-2x-y) = e^{y-x}(x^2-4x+xy-2y+2)$

$\dfrac{\partial^2 f}{\partial y\partial x} = e^{y-x}(2x+y-x^2-xy) + e^{y-x}(1-x) = e^{y-x}(x+y-x^2-xy+1)$

$\dfrac{\partial^2 f}{\partial y^2} = xe^{y-x}(x+y+1) + xe^{y-x} = xe^{y-x}(x+y+2)$

$\dfrac{\partial^2 f}{\partial x\partial y} = e^{y-x}(x+y+1) - xe^{y-x}(x+y+1) + xe^{y-x}\cancel{y} = e^{y-x}(x+y+1-x^2-xy\cancel{-x}+\cancel{x}) = e^{y-x}(x+y+1-x^2-x$

$$H(x,y) = \begin{bmatrix} e^{y-x}(x^2-4x+xy-2y+2) & e^{y-x}(x+y-x^2-xy+1) \\ e^{y-x}(x+y-x^2-xy+1) & xe^{y-x}(x+y+2) \end{bmatrix}$$

$\det H(x,y) = e^{y-x}(x^2-4x+xy-2y+2)\cdot xe^{y-x}(x+y+2) - \left[e^{y-x}(x+y-x^2-xy+1)\right]^2$

$\det H(0,0) = 1\cancel{(2)\cdot 0}\ 1-2 - [1(1)]^2 = -1 < 0$ $\quad$ SELLA

$\det H(\frac{1}{2}, -\frac{3}{2}) = e^{-2}\left(\frac{1}{4}-2-\frac{3}{4}+3+2\right)\cdot\frac{1}{2}e^{-2}\left(\frac{1}{2}-\frac{3}{2}+2\right) - \left[e^{-2}\left(\frac{1}{2}-\frac{3}{2}-\frac{1}{4}+\frac{3}{4}+1\right)\right]^2 =$

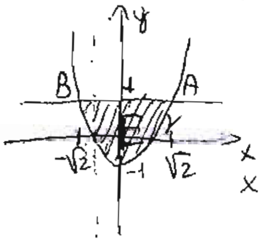
$= e^{-2}\cdot\dfrac{\overset{5}{\cancel{10}}}{4}\cdot\dfrac{1}{\cancel{2}}e^{-2}\cdot\cancel{1} - \left[e^{-2}\cdot\dfrac{2}{4}\right]^2 = \dfrac{5}{4}e^{-4} - \dfrac{1}{4}e^{-4} = e^{-4} > 0$

$\dfrac{\partial^2 f}{\partial x^2}(\frac{1}{2}, -\frac{3}{2}) = e^{-2}\left(\frac{1}{4}-2-\frac{3}{4}+3+2\right) = e^{-2}\cdot\dfrac{5}{2} > 0$ $\quad$ MINIMO $\quad \min\limits_{\mathbb{R}^2} f(x,y) = f(\frac{1}{2}, -\frac{3}{2}) = -\frac{1}{2}e$

3.5.10

iii) $f(x,y) = x^2+2y$ $\quad \dfrac{\partial f}{\partial x} = 2x$ $\quad \dfrac{\partial f}{\partial y} = 2$ $\quad E = \{(x,y)\in\mathbb{R}^2 : x^2-1 \le y \le 1\}$

$\begin{cases} 2x = 0 \\ 2 = 0 \end{cases}$ IMP. $\nexists$ PUNTI STAZIONARI $\Rightarrow$ studio il bordo



$\begin{cases} y = x^2-1 \\ y = 1 \end{cases}$

$x^2 = 2 \quad x = \pm\sqrt{2}$

$AB \begin{cases} x=t \\ y=1 \end{cases} \quad t\in[-\sqrt{2},\sqrt{2}] \qquad f(A)=f(\sqrt{2},1)=2+2=4 \qquad \underset{B}{f}(-\sqrt{2},1)=2+2=4$ 23/04/08

Studio $f(x,y)$ su una curva. $f(x(t),y(t))=g(t)$ $\quad g_1(t)=t^2+2 \quad t\in[-\sqrt{2},\sqrt{2}]$

$g_1'(t)=2t \quad 2t=0 \quad t=0\in[-\sqrt{2},\sqrt{2}]$ se $\overset{t=0}{\underset{\substack{x=0\\y=1}}{\Downarrow}}$ $f(0,1)=2$

$$\gamma \begin{cases} x=t \\ y=t^2-1 \end{cases} t\in[-\sqrt{2},\sqrt{2}] \qquad g_2(t)=t^2+2t^2-2=3t^2-2 \quad g_2'(t)=6t \quad g_2'(t)=0 \quad 6t=0 \quad t=0$$

$$f(0,-1)=-2$$

$$\max_E{}^{ASS} f(x,y) = \max\{f(A),f(B),f(0,1),f(0,-1)\}=4=f(\sqrt{2},1)=f(-\sqrt{2},1)$$

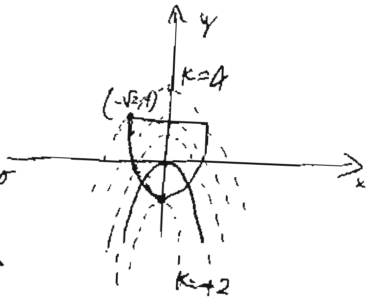
$$\min_E{}^{ASS} f(x,y) = \min\{f(A),f(B),f(0,1),f(0,-1)\}=-2=f(0,-1)$$

Con gli insiemi di livello diventa

$x^2+2y=K \qquad y=-\frac{x^2}{2}+\frac{K}{2}$

La parabola con $K=-2$ mi dà il minimo

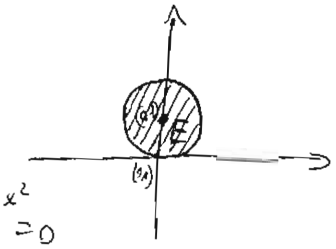
$y=-\frac{x^2}{2}+\frac{K}{2} \qquad 1=-\frac{2}{2}+\frac{K}{2} \qquad K=4$ max.



Come mi aspettavo

(V) $f(x,y)=y\,e^{-x^2} \qquad E=\{(x,y)\in\mathbb{R}^2: x^2+y^2-2y\leq 0\}$

Per T. di Weierstrass $\exists$ max e min. $\overset{\Downarrow}{\text{cfr}}$ centrata in $(0,1)$



$\frac{\partial f}{\partial x}=y e^{-x^2}\cdot(-2x)=-2xye^{-x^2} \qquad \frac{\partial f}{\partial y}=e^{-x^2} \qquad \begin{cases}-2xye^{-x^2}=0\\ e^{-x^2}=0\end{cases}$ $\nexists$ punti critici

Studio il bordo

$\begin{cases} x=\cos t \\ y=1+\operatorname{sen} t\end{cases} t\in[0,2\pi] \qquad g(t)=(1+\operatorname{sen}t)\cdot e^{-\cos^2 t} \quad g'(t)=\cos t\, e^{-\cos^2 t}+(1+\operatorname{sen}t)\cdot e^{-\cos^2 t}\cdot 2\cos t \operatorname{sen}t$

$$=\underbrace{e^{-\cos^2 t}\cos t\,(1+(1+\operatorname{sen}t)(2\operatorname{sen}t))=0}$$

$\cos t\, e^{-\cos^2 t}(1+2\operatorname{sen}t+2\operatorname{sen}^2 t)=0 \qquad \cos t=0$ (esp. mai 0) $\quad t=\frac{\pi}{2} \vee t=\frac{3}{2}\pi$

$2\operatorname{sen}^2 t+2\operatorname{sen}t+1=0 \quad \operatorname{sen}t_{1,2}=\frac{-1\pm\sqrt{1-2}}{2}<0$ MAI

$\overset{\Downarrow}{t=\frac{\pi}{2}}$ e $t=\frac{3}{2}\pi$

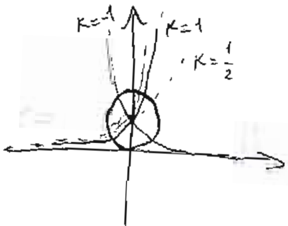
$\longrightarrow f(0,0)=0$ min ASS.

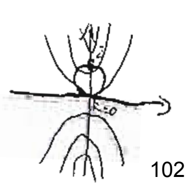
$\Rightarrow f(0,2)=2$ max ASS.

INS. DI LIVELLO

$y=Ke^{x^2}$ $\quad K=1$

$y'=2xe^{x^2}\geq 0$ per $x\geq 0$





3.5.17

③

$f(x,y) = e^{-x^2-y} - y \qquad \frac{\partial f}{\partial x} = e^{-x^2-y} \cdot (-2x) \qquad \frac{\partial f}{\partial y} = -e^{-x^2-y} - 1$

i) $\begin{cases} e^{-x^2-y} \cdot (-2x) = 0 \\ -e^{-x^2-y} - 1 = 0 \end{cases} \qquad \begin{cases} x = 0 \\ +e^{-x^2-y} = -1 \end{cases}$ $\nexists$ punti stazionari

ii) inf, sup : ?

$f(0,y) = e^{-y} - y \xrightarrow{y \to \pm\infty} \mp\infty$ inf $= -\infty$

sup $= +\infty$

iii) Trovare max e min assoluti su $\mathbb{R}^2$ $\nexists$ non ci sono punti critici

TEMI D'ESAME 15/09/2006

$f(x,y) = (x^2-1)^2 \cdot (y-3)^2 + \frac{1}{2} y^2$

a) Determinate gli eventuali punti stazionari di f in $\mathbb{R}^2$ e studiatene la natura

b) Dopo averne giustificato l'esistenza determinate max e min assoluti in

$E = \{(x,y) \in \mathbb{R}^2 : 0 \le |x| \le \sqrt{2},\ |y| \le 2\}$

a) $\frac{\partial f}{\partial x} = 2(x^2-1) \cdot 2x(y-3)^2 \qquad \frac{\partial f}{\partial y} = (x^2-1)^2 \cdot 2(y-3) + y$

$\begin{cases} 4x(x^2-1)(y-3)^2 = 0 \\ (x^2-1)^2 \cdot 2(y-3) + y = 0 \end{cases}$ $x = 0 \lor x = \pm 1 \lor y = 3$

$\begin{cases} x = 0 \\ 2y - 6 + y = 0 \end{cases} \begin{cases} x = 0 \\ y = 2 \end{cases} A = (0,2) \qquad \begin{cases} x = \pm 1 \\ y = 0 \end{cases} \begin{matrix} B = (1,0) \\ C = (-1,0) \end{matrix}$

$\begin{cases} y = 3 \\ 3 = 0 \end{cases}$ IMP. $\frac{\partial^2 f}{\partial x^2} = 4(x^2-1)(y-3)^2 + 4x \cdot 2x \cdot (y-3)^2 = (y-3)^2 \cdot (4x^2 - 4 + 8x^2) = (y-3)^2 (12x^2 - 4)$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 8x(x^2-1)(y-3) \qquad \frac{\partial^2 f}{\partial y^2} = (x^2-1)^2 \cdot 2 + 1 =$ $H(x,y) = \begin{bmatrix} (y-3)^2(12x^2-4) & 8x(x^2-1)(y-3) \\ 8x(x^2-1)(y-3) & 1 + 2(x^2-1)^2 \end{bmatrix}$

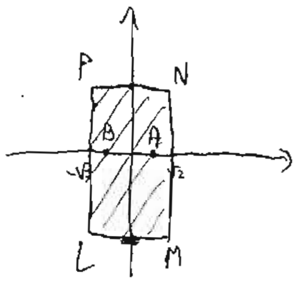
$\det H(x,y) = (y-3)^2 (12x^2-4) \left[1 + 2(x^2-1)^2\right] - \left[8x(x^2-1)(y-3)\right]^2$

$\det H(A(0,2)) = 1 \cdot (-4) \cdot [1+2] - 0 = -12 < 0$ SELLA $f(0,2) = 3$

$\det H(B) = 9 \cdot 8 \cdot [1+0] - [\cancel{8 \cdot 0 \cdot (-3)}]^2 = 72 > 0 \qquad \frac{\partial^2 f}{\partial x^2}(1,0) = 9 \cdot 8 > 0$ MINIMO $f(1,0) = 0$

$\det H(C) = 9 \cdot 8 [1] - (\cancel{-8 \cdot (0) \cdot}) = 72 > 0 \qquad \frac{\partial^2 f}{\partial x^2}(-1,0) = 9 \cdot 8 > 0$ MINIMO $f(-1,0) = 0$

b)



$|x| \geq 0 \ \forall x$

$|x| \leq \sqrt{2} \quad -\sqrt{2} \leq x \leq \sqrt{2}$

$|y| \leq 2 \quad -2 \leq y \leq 2$

Per il T. di Weierstrass $\exists$ max e min

$f(A)=0 \quad f(B)=0$

$f(N) = f(\sqrt{2},2) = 3 \quad f(P) = f(-\sqrt{2},2) = 3 \quad f(L) = f(-\sqrt{2},-2) = 27$

$f(M) = f(\sqrt{2},-2) = 27$

$\overline{PN} = \begin{cases} x = t \\ y = 2 \end{cases} \quad t \in [-\sqrt{2},\sqrt{2}] \quad g_1(t) = (t^2-1)^2 \cdot 1 + 2 \quad g_1'(t) = 2(t^2-1) \cdot 2t = (2t^2-2) \cdot 2t) = 0 \quad \begin{matrix} t = \pm 1 \\ t = 0 \end{matrix}$

$f(\pm 1, 2) = 2 \qquad f(0,2) = 3$

$\overline{MN} - \begin{cases} x = \sqrt{2} \\ y = t \end{cases} \quad t \in [-2,2] \quad g_2(t) = (t-3)^2 + \frac{1}{2}t^2 \quad g_2'(t) = 2(t-3) + t = 3t - 6 = 0 \quad t = 2$ GIÀ STUDIATO (ESTREMO)

$\overline{LM} \begin{cases} x = t \\ y = -2 \end{cases} \quad t \in [-\sqrt{2},\sqrt{2}] \quad g_3(t) = (t^2-1)^2 \cdot 25 + 2 \quad g_3'(t) = 2(t^2-1) \cdot 25 \cdot 2t = 0 \quad \begin{matrix} t = \pm 1 \\ t = 0 \end{matrix}$

$f(\pm 1, -2) = 2 \qquad f(0,-2) = 27$

$\overline{LP} \begin{cases} x = -\sqrt{2} \\ y = t \end{cases} \quad t \in [-2,2] \quad g_4(t) = (t-3)^2 + \frac{1}{2}t^2 \quad g_4'(t) = 2(t-3) + t = 3t - 6 = 0 \quad t = 2$ GIÀ STUDIATO (ESTREMO)

VALORE + PICCOLO : $0 \Rightarrow$ min ASS $\underset{E}{f(x,y)} = 0 = f(-1,0) = f(1,0)$

VALORE + GRANDE : $27 \Rightarrow$ max ASS $\underset{E}{f(x,y)} = 27 = f(-\sqrt{2},-2) = f(\sqrt{2},-2) = f(0,-2)$

ESERCITAZIONE

24/04/0

①
$$f(x,y) = x^2 - 2xy^3 + 3y^2 + 1$$

(a) punti stazionari e natura

(b) max e min assoluti sul quadrato $(0,0)$ $(1,0)$ $(1,1)$ $(0,1)$

$$\frac{\partial f}{\partial x} = 2x - 2y^3 \qquad \frac{\partial f}{\partial y} = -6xy^2 + 6y \qquad \begin{cases} \cancel{2}x - \cancel{2}y^3 = 0 \\ \cancel{6}y - \cancel{6}xy^2 = 0 \end{cases} \qquad \begin{cases} x = y^3 \\ y - y^5 = 0 \end{cases}$$

$$\begin{cases} y(1 - y^4) = 0 \\ x = y^3 \end{cases} \qquad \begin{cases} y = 0 \\ x = 0 \end{cases} \qquad \begin{cases} y = +1 \\ x = +1 \end{cases} \qquad \begin{cases} y = -1 \\ x = -1 \end{cases} \qquad A(0,0) \quad B(1,1) \quad C(-1,-1)$$

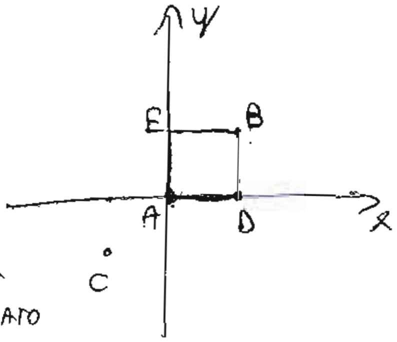
$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y \partial x} = -6y^2 \qquad \frac{\partial^2 f}{\partial y^2} = -12xy + 6 \qquad H(x,y) = \begin{pmatrix} 2 & -6y^2 \\ -6y^2 & -12xy+6 \end{pmatrix}$$

$$\det H(x,y) = 2(-12xy+6) - (-6y^2)^2 = 12 - 24xy - 36y^4$$

$\det H(A) = 12 > 0$ $\quad \frac{\partial^2 f}{\partial x^2}(A) = 2 > 0$ $\quad$ A è punto di minimo $f(0,0) = 1$

$\det H(B) = -48 < 0$ $\quad$ B è punto di sella

$\det H(C) = -48 < 0$ $\quad$ C è punto di sella



$\overline{AD}$ $\begin{cases} x = t \\ y = 0 \end{cases}$ $t \in [0,1]$ $\quad \underline{f(A) = 1} \quad \underline{f(D) = 2}$

$g_1(t) = t^2 + 1 \qquad g_1'(t) = 2t \qquad 2t = 0 \quad t = 0 \quad \begin{cases} x = 0 \\ y = 0 \end{cases}$ GIÀ TROVATO

$\overline{DB}$ $\begin{cases} x = 1 \\ y = t \end{cases}$ $t \in [0,1]$ $\quad \underline{f(B) = 3} \quad g_2(t) = 1 - 2t^3 + 3t^2 + 1 \quad g_2'(t) = 6t - 6t^2$

$6t(1-t) = 0 \qquad t = 0 \begin{cases} x = 1 \\ y = 0 \end{cases} \qquad t = 1 \begin{cases} x = 1 \\ y = 1 \end{cases} \qquad \underline{f(1,0) = 2}$

$\overline{BE}$ $\begin{cases} x = t \\ y = 1 \end{cases}$ $t \in [0,1]$ $\quad \underline{f(E) = 4} \qquad g_3(t) = t^2 - 2t + 4 \quad g_3'(t) = 2t - 2 \quad 2t - 2 = 0 \quad t = 1$

$\begin{cases} x = 1 \\ y = 1 \end{cases}$ GIÀ TROVATI

$\overline{AE}$ $\begin{cases} x=0 \\ y=t \end{cases}$ $t\in[0,1]$ $g_4(t)=3t^2+1$ $g_4'(t)=6t$ $6t=0$ $t=0$ $\begin{cases} x=0 \\ y=0 \end{cases}$ GIÀ TROVATO

$\min\limits_{A} f(x,y) = f(0,0) = 1$ $\max\limits_{A} f(x,y) = f(0,1) = 4$

② $f(x,y) = (x^2-2y^2)e^{x-y}$

a) punti stazionari e natura

b) $\lim\limits_{\|(x,y)\|\to\infty} f(x,y) = +\infty$ VERO O FALSO?

c) la funzione è limitata inf

VERO O FALSO?

$$\frac{\partial f}{\partial x} = 2x\,e^{x-y} + (x^2-2y^2)e^{x-y}$$

$$\frac{\partial f}{\partial y} = -4y\,e^{x-y} - (x^2-2y^2)e^{x-y}$$

$$\begin{cases} \cancel{e^{x-y}}(x^2-2y^2+2x)=0 \\ \cancel{e^{x-y}}(-4y-x^2+2y^2)=0 \end{cases} \qquad e^{x-y}>0\ \forall x$$

$$\begin{cases} 2x-4y=0 \\ \dots \end{cases} \quad \begin{cases} x=2y \\ -4y-4y^2+2y^2=0 \end{cases} \quad \begin{cases} x=2y \\ 2y+y^2=0 \end{cases} \quad \begin{cases} y(y+2)=0 \\ x=2y \end{cases} \quad \begin{cases} y=0 \\ x=0 \end{cases} \quad \begin{cases} y=-2 \\ x=-4 \end{cases}$$

$A(0,0)$ $B(-4,-2)$

$$\frac{\partial^2 f}{\partial x^2} = e^{x-y}(x^2-2y^2+2x) + e^{x-y}(2x+2) \qquad \frac{\partial^2 f}{\partial y\partial x} = -e^{x-y}(x^2-2y^2+2x) + e^{x-y}(-4y)$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{x-y}(2y^2-4y-x^2) + e^{x-y}(4y-4)$$

$$H = \begin{pmatrix} e^{x-y}(x^2-2y^2+4x+2); & e^{x-y}(-x^2+2y^2-2x-4 \\ e^{x-y}(-x^2+2y^2-2x-4y); & e^{x-y}(-2y^2+8y+x^2-4 \end{pmatrix}$$

$$\det H(x,y) = e^{x-y}(x^2-2y^2+4x+2)e^{x-y}(-2y^2+8y+x^2-4) + -\left[e^{x-y}(-x^2+2y^2-2x-4y)\right]^2$$

$\det H(0,0) = 1\cdot2\cdot1\cdot(-4) - 0 = -8 < 0$ SELLA

$\det H(-4,-2) = e^{-2}(\cancel{16}-8-\cancel{16}+2)e^{-2}(-8-\cancel{16}+\cancel{16}-4) - e^{-4}(-\cancel{16}+8+\cancel{8}+8)^2 =$

$= e^{-4}(-6)(-12) - e^{-4}\cdot 64 = e^{-4}(8) > 0$ $\frac{\partial^2 f}{\partial x^2}(-4,-2) = e^{-2}(0) + e^{-2}(-8+2) = -6e^{-2}$

MASSIMO

) $f(0,y)=(-2y^2)\cdot e^{-y}$ $\lim_{y\to+\infty} = -\infty\cdot 0 = 0$ FALSO

c) inf f? $f(x,0)=x\cdot e^x$ $\lim_{x\to-\infty} f(x)=0$

FALSO

$f(0,y)=-2y^2\cdot e^{-y}$ $\lim_{y\to-\infty} = -\infty$

③ $f(x,y)=x^2+y^2-2y+1$

(a) punti stazionari e natura

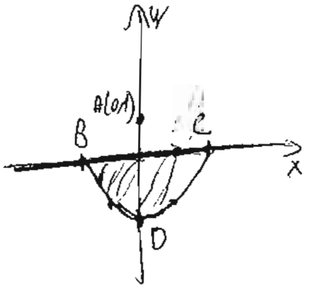
(b) max e min su $A=\{(x,y)\in\mathbb{R}^2 : 0\geq y\geq -\sqrt{4-x^2}\}$

(c) Se P_M (P_m) punto di massimo (minimo) di f su A cade su ∂A e se in questo punto la frontiera ammette una parametrizzazione regolare $\varphi(t)$ cosa si può dire del prodotto scalare $\langle\nabla f(P_M), \varphi'(t_M)\rangle$ dove $\varphi(t_M)=P_M$?

$$\frac{\partial f}{\partial x}=2x \qquad \frac{\partial f}{\partial y}=2y-2 \qquad \begin{cases}2x=0\\2y-2=0\end{cases} \qquad \begin{cases}x=0\\y=1\end{cases} \qquad H\begin{pmatrix}2 & ; & 0\\0 & ; & 2\end{pmatrix}$$

$\det H(0,1) = 4>0$ $\frac{\partial^2 f}{\partial x^2}(0,1)=2>0$ $A(0,1)$ è punto di minimo.

(b) $y=0$



$y=-\sqrt{4-x^2}$ $y^2=-4+x^2$

$4-x^2\geq 0$
$x^2\leq 4$
$-2\leq x\leq 2$

$x^2-y^2=4$

MINIMO ASSOLUTO

$f(A)=f(0,1)=0$ $f(C)=f(1,0)=2$

$f(B)=f(-1,0)=2$ $f(D)=f(0,-2)=9$

$\overline{BC}$ $\begin{cases}x=t\\y=0\end{cases}$ $t\in[-2,2]$ $g_1(t)=t^2+1$ $g_1'(t)=2t$ $2t=0$ t_0 $\begin{cases}x=0\\y=0\end{cases}$ $f(0,0)=1$

$\overset{\frown}{BC}$ $\begin{cases}x=2\cos t\\y=2\operatorname{sen}t\end{cases}$ $t\in[\pi,2\pi]$ ~~$g_2(t)=t^2-4+t^2+2\sqrt{4-t^2}+1=2t^2+2\sqrt{4-t^2}+-3$~~

~~$2t^2+2\sqrt{4-t^2}-3=0$~~ ~~$2\sqrt{4-t^2}=3-2t^2$ $4(4-t^2)=9-12t^2+4t^4$~~

~~$4t^4-8t^2-7=0$~~ ~~$t_{1,2}=\frac{4\pm\sqrt{16+28}}{4}=\frac{4\pm\sqrt{44}}{4}$~~

$g_2(t)=4\cos^2 t+4\operatorname{sen}^2 t-4\operatorname{sen}t+1 = 4+1-4\operatorname{sen}t=5-4\operatorname{sen}t$ $g_2'(t)=-4\cos t$ $-4\cos t=0$

$t=\frac{\pi}{2}$ ~~o~~ $t=\frac{3\pi}{2}\in[\pi,2\pi]$ $\begin{cases}x=0\\y=-2\end{cases}$ $f(0,-2)=9$ MASSIMO ASSOLUTO SU A

④ $f(x,y) = y - x^2$ Trovare max e min di f su $A = \{(x,y) \in \mathbb{R}^2 : |x| \leq |y| \leq 2\}$

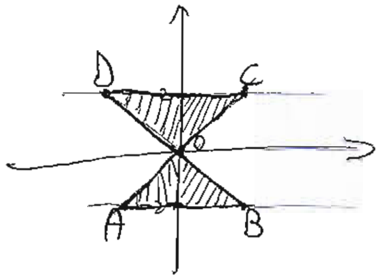
$|x| \leq |y|$

$y \geq x$ se $x>0, y>0$

$y \geq -x$ se $x<0, y>0$

$y \leq x$ se $x<0, y<0$

$y \leq -x$ se $x>0, y<0$



$|y| \leq 2 \qquad -2 \leq y \leq 2$

$A=(-2,-2) \quad B=(2,-2) \quad C=(2,2) \quad D=(-2,2)$

$f(-2,-2) = -6 \quad f(2,2) = -2$

$f(2,-2) = -4 \quad f(-2,2) = -6$

$\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial f}{\partial y} = 1 \qquad \begin{cases} -2x=0 \\ 1=0 \end{cases}$ ∄ punti staz.

AB $\begin{cases} x=t \\ y=-2 \end{cases}$ $t\in[-2,2]$ $\quad g_1(t) = -2-t^2 \quad -t^2 = 2 \quad t^2 = -2$ MAI

BD $\begin{cases} x=t \\ y=-t \end{cases}$ $t \in [-2,2] \quad g_2(t) = -t-t^2 \quad -(t^2+t)=0 \quad t(t+1)=0$

$t=0 \begin{cases} x=0 \\ y=0 \end{cases} \quad f(0,0)=0$

$t=-1 \begin{cases} x=-1 \\ y=1 \end{cases} \quad f(-1,1)=0$

AC $\begin{cases} x=t \\ y=t \end{cases}$ $t\in[-2,2] \quad g_3(t) = t-t^2 \quad t(1-t)=0$

$t=0 \begin{cases} x=0 \\ y=0 \end{cases}$

$t=1 \begin{cases} x=1 \\ y=1 \end{cases} \quad f(1,1)=0$

CD $\begin{cases} x=t \\ y=2 \end{cases}$ $t\in[-2,2] \quad g_4(t) = 2-t^2 \quad -2t=0 \quad t=0 \quad \begin{cases} x=0 \\ y=2 \end{cases}$ $\quad \cancel{\begin{cases} x=-\sqrt{2} \\ y=2 \end{cases}}$

$f(0,2)=2$

~~$f(\sqrt{2},2)=0$~~

$\text{MAX ASS}_A\ f = 2 = f(0,2) = $ ~~$f(-1,1) = f(1,1) = f(\sqrt{2},2) = f(-\sqrt{2},2)$~~

$\text{MIN ASS}_A\ f = -6 = f(-2,-2) = f(-2,+2)$

ESAME 5/5/2007

$f(x,y) = xy(x+y+3)$ 1) punti stazionari su $\mathbb{R}^2$ 2) max e min ass in $E = \{(x,y) \in \mathbb{R}^2 : x \geq 0;\ 0 \leq y \leq 4-x\}$

$$\frac{\partial f}{\partial x} = y(x+y+3) + xy = y(2x+y+3) = 0 \quad \begin{cases} y = 0 \\ 2x+y+3 = 0 \end{cases}$$

$$\frac{\partial f}{\partial y} = x(x+y+3) + xy = x(x+2y+3) = 0$$

$$\begin{cases} y = 0 \\ x(x+3) = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ o } \begin{cases} x = -3 \\ y = 0 \end{cases} \qquad \begin{cases} y = -2x-3 \\ x(x-4x-6+3) = 0 \end{cases} \begin{cases} y = -2x-3 \\ x(-3x-3) = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = -3 \end{cases} , \begin{cases} x = -1 \\ y = -1 \end{cases}$$

$O(0,0) \quad A(-3,0) \quad B(0,-3) \quad C(-1,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x+y+3+y = 2x+2y+3 = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y^2} = 2x$$

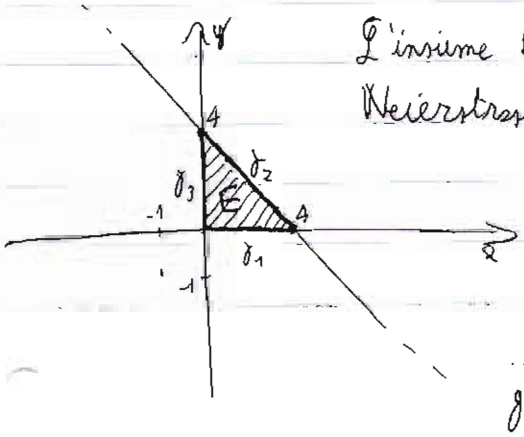
$$H = \begin{pmatrix} 2y & 2x+2y+3 \\ 2x+2y+3 & 2x \end{pmatrix} \qquad \det H = 4xy - (2x+2y+3)^2$$

$\det H(0,0) = -9 < 0$ SELLA $\qquad \det H(-3,0) = \overset{0}{[illegible]} - (-6+3)^2 = \overset{-9}{[illegible]} < 0$ SELLA

$\det H(0,-3) =$ [illegible] $-9 < 0$ SELLA $\qquad \det H(-1,-1) = 4 - (-2-2+3)^2 = 4-1 = 3 > 0$

$$\frac{\partial^2 f}{\partial x^2}(-1,-1) = -2 < 0 \quad \text{MASSIMO RELATIVO} \quad f(-1,-1) = 1$$

2) L'insieme E è chiuso e limitato, f è continua, per il T. di Weierstrass esistono max e min assoluti



$$\gamma_1 : \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in [0,4]$$

[illegible] $=$ $f(0,0) = 0$

$f(0,4) = 0$

$f(4,0) = 0$

$\gamma_1(t) = 0$

$\gamma_3 \begin{cases} x=0 \\ y=t \end{cases} t\in[0,4] \quad g_3(t)=0$ $\qquad \gamma_2 \begin{cases} x=t \\ y=4-t \end{cases} t\in[0,4] \quad g_2(t)=t(4-t)(\cancel{t}+4-\cancel{t}+3)=-7t^2+28t$

$g_2'(t)=-14t+28=0 \quad 14t=28 \quad t=2 \quad f(2,2)=28$

min ASS $\underset{E}{f}(x,y)=0=f(\gamma_1)=f(\gamma_3)$ ...

max ASS $\underset{E}{f}(x,y)=28=f(2,2)$

1/9/06

$f(x,y)=-x^4+4x^2y-4y^2-2x^2+\frac{4}{3}y^3$

1) pti stazionari

2) $E=\{(x,y)\in\mathbb{R}^2: 0\le x\le 2;\ 0\le y\le x\}$

$\frac{\partial f}{\partial x}=-4x^3+8xy-4x \qquad \frac{\partial f}{\partial y}=4x^2-8y+4y^2$

$\begin{cases} x(-4x^2+8y-4)=0 \\ 4x^2-8y+4y^2=0 \end{cases} \quad \begin{cases} x=0 \\ 4y^2-8y=0 \end{cases} \quad \begin{cases} x=0 \\ y=0 \end{cases} \text{ o } \begin{cases} x=0 \\ y=2 \end{cases} \quad O(0,0) \quad A(0,2)$

$\begin{cases} -4x^2+8y-4=0 \\ 4x^2-8y+4y^2=0 \end{cases}$ $\quad$ " " $4y^2-4=0$ $\quad \begin{cases} y^2=1 \\ 4x^2-8y+4y^2=0 \end{cases} \quad \begin{cases} y=1 \\ x^2=1 \end{cases} \quad \begin{cases} y=1 \\ x=1 \end{cases} \text{ o } \begin{cases} x=-1 \\ y=1 \end{cases} \quad B(1,1) \quad C(-1,1)$

$\begin{cases} y=-1 \\ x^2=-3 \end{cases}$ MAI

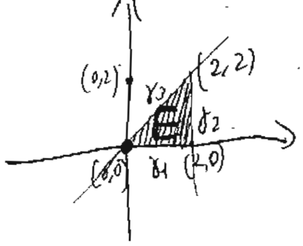
$\frac{\partial^2 f}{\partial x^2}=-12x^2+8y-4 \qquad \frac{\partial^2 f}{\partial y\partial x}=8x \qquad \frac{\partial^2 f}{\partial y^2}=-8+8y$

$H=\begin{pmatrix} -12x^2+8y-4 & 8x \\ 8x & 8y-8 \end{pmatrix} \qquad \det H=(-12x^2+8y-4)(8y-8)-64x^2$

$\det H(0,0)=32>0 \qquad \frac{\partial^2 f}{\partial x^2}(0,0)=-4<0$ MASSIMO LOCALE $f(0,0)=0$

$\text{et } H(0,2)=12\cdot 8>0 \qquad \frac{\partial^2 f}{\partial x^2}(0,2)=12>0$ MINIMO LOCALE $f(0,2)=-16+\frac{32}{3}=-\frac{16}{3}$

$\text{et } H(1,1)=-64<0$ SELLA $\qquad \det H(-1,1)=-64<0$ SELLA



$f(0,0)=0 \leftarrow$

$f(2,2)=-\cancel{16}+\cancel{32}-\cancel{16}-8+\frac{32}{3}=\frac{8}{3} \leftarrow$

$f(2,0)=-16-8=-24 \leftarrow$

$\gamma_1 \begin{cases} x=t \\ y=0 \end{cases} t\in[0,2] \quad g_1(t)=-t^4-2t^2$

$g_1'(t)=-4t^3-4t \quad -4(t^3+t)=0 \quad t(t^2+1)=0 \quad t=0 \quad f(0,0)$ già calcolata

$t^2+1=0$ MAI

$\gamma_2 \begin{cases} x=2 \\ y=t \end{cases} \quad t\in[0,2] \qquad g_2(t) = -16+16t-4t^2-8+\frac{4}{3}t^3 \qquad g_2'(t) = 4t^2-8t+16$

$g_2'(t)=0 \qquad t^2-2t+4=0 \qquad t_{1,2} = \frac{1\pm\sqrt{1-4}}{\cancel{2}}$ MAI

$\gamma_3 \begin{cases} x=t \\ y=t \end{cases} \quad t\in[0,2] \qquad g_3(t) = -t^4+4t^3-4t^2-2t^2+\frac{4}{3}t^3 \qquad g_3'(t) = -4t^3+12t^2+4t^2-12t$

$g_3'(t)=0 \quad -4t(t^2-4t+3)=0 \qquad t=0 \quad \begin{cases} x=0 \\ y=0 \end{cases}$ già studiato

$t^2-4t+3=0 \quad (t-1)(t-3)=0 \qquad t=1 \quad f(1,1) = -\frac{5}{3} \Leftarrow$

$t=3 \quad f(3,3) \notin [0,2]$

MAX ASS $f_E(x,y) = \frac{8}{3} = f(2,2)$ MIN ASS $f_E(x,y) = -24 = f(2,0)$

1/9/06

$f(x,y) = y(4x^2-1)e^{-\frac{y^2}{2}} = (4x^2y-y)e^{-\frac{y^2}{2}}$

$\frac{\partial f}{\partial x} = 8xye^{-\frac{y^2}{2}} \qquad \frac{\partial f}{\partial y} = (4x^2-1)e^{-\frac{y^2}{2}} + -y(4x^2y-y)e^{-\frac{y^2}{2}} = e^{-\frac{y^2}{2}}(4x^2-1-4x^2y^2+y^2)$

$\begin{cases} 8xye^{-\frac{y^2}{2}} = 0 \\ e^{-\frac{y^2}{2}}(4x^2-1-4x^2y^2+y^2)=0 \end{cases} \qquad \begin{cases} x=0 \\ (y^2-1)e^{-\frac{y^2}{2}}=0 \end{cases} \qquad \begin{cases} x=0 \\ y=\pm 1 \end{cases} \qquad \begin{matrix} A(0,1) \\ B(0,-1) \end{matrix}$

$\begin{cases} \cancel{x}y=0 \\ 4x^2-1=0 \end{cases} \qquad \begin{cases} y=0 \\ x=\pm\frac{1}{2} \end{cases} \qquad \begin{matrix} C\left(\frac{1}{2},0\right) \\ D\left(-\frac{1}{2},0\right) \end{matrix}$

$\frac{\partial^2 f}{\partial x^2} = 8ye^{-\frac{y^2}{2}} \qquad \frac{\partial^2 f}{\partial x \partial y} = 8xe^{-\frac{y^2}{2}} - 8y^2xe^{-\frac{y^2}{2}} = 8xe^{-\frac{y^2}{2}}(1-y^2)$

$\frac{\partial^2 f}{\partial y^2} = e^{-\frac{y^2}{2}}(-y)\cdot(4x^2-1-4x^2y^2+y^2)+e^{-\frac{y^2}{2}}(2y-8x^2y) =$

$= e^{-\frac{y^2}{2}}(-4x^2y+y+4x^2y^3-y^3+2y-8x^2y) = e^{-\frac{y^2}{2}}(-12x^2y+3y+4x^2y^3-y^3)$

$$H = \begin{pmatrix} 8ye^{-\frac{y^2}{2}} & 8xe^{-\frac{y^2}{2}}(1-y^2) \\ 8xe^{-\frac{y^2}{2}}(1-y^2) & ye^{-\frac{y^2}{2}}(-12x^2+3+4x^2y^2-y^2) \end{pmatrix}$$

$\det H(0,1) = 8e^{-\frac{1}{2}}\cdot 2e^{-\frac{1}{2}} - 0 = 16e^{-1} = \frac{16}{e} > 0$

$\frac{\partial^2 f}{\partial x^2}(0,1) = 8e^{-\frac{1}{2}} > 0$ MINIMO LOCALE

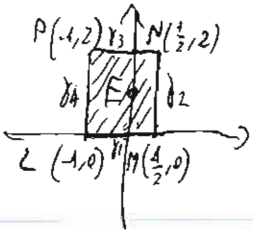
$f(0,1) = -e^{-\frac{1}{2}}$

$\det H(0,-1) = -8e^{-\frac{1}{2}}\cdot\left(-2e^{-\frac{1}{2}}\right) - 0 = 16e^{-1} > 0 \qquad \frac{\partial^2 f}{\partial x^2}(0,-1) = -8e^{-\frac{1}{2}} < 0$ MASSIMO LOCALE

$f(0,-1) = e^{-\frac{1}{2}}$

$\det H\left(\frac{1}{2},0\right) = -\left(\cancel{8}^{4}\cdot\frac{1}{2}\cdot e^{0}\cdot 1\right)^2 = -16 < 0$ SELLA $\quad \det H\left(-\frac{1}{2},0\right) = -\left(8\cdot\left(-\frac{1}{2}\right)\cdot 1\cdot 1\right)^2 = -16 < 0$ SELLA

$E = \left\{ (x,y) \in \mathbb{R}^2 : -1 \leq x \leq \frac{1}{2}, 0 \leq y \leq 2 \right\}$



$f(0,1) = -\frac{1}{\sqrt{e}}$ $\quad f(-1,0) = 0 \quad f\left(\frac{1}{2},0\right) = 0$

$f(-1,2) = \frac{6}{e^2}$

$\gamma_1 \begin{cases} x = t \\ y = 0 \end{cases} \quad t \in \left[-1, \frac{1}{2}\right] \quad g_1(t) = 0 \qquad \gamma_2 \begin{cases} x = \frac{1}{2} \\ y = t \end{cases} \quad t \in [0,2] \quad g_2(t) = 0$

$\gamma_3 \begin{cases} x = t \\ y = 2 \end{cases} \quad t \in \left[-1, \frac{1}{2}\right] \quad g_3(t) = 2(4t^2 - 1)e^{-2} \quad g_3'(t) = 2e^{-2} \cdot 8t = 0 \quad t = 0 \quad f(0,2) = \frac{-2}{e^2}$

$\gamma_4 \begin{cases} x = -1 \\ y = t \end{cases} \quad t \in [0,2] \quad g_4(t) = 3te^{-\frac{t^2}{2}} \quad g_4'(t) = 3\left(e^{-\frac{t^2}{2}} + te^{-\frac{t^2}{2}} \cdot (-t)\right) = 0 \quad 3e^{-\frac{t^2}{2}}(1 - t^2) = 0 \quad t = +1$
$\quad t = -1 \notin [\,]$

$f(-1,1) = \frac{3}{\sqrt{e}}$ $\qquad$ MAX ASS $\underset{E}{f(x,y)} = \frac{3}{\sqrt{e}} = f(-1,1)$ $\qquad$ MIN ASS $\underset{E}{f(x,y)} = -\frac{1}{\sqrt{e}} = f(0,1)$

ESERCITAZIONE

8/5/08

①

① a) $y' = -\frac{y}{x} + senx \qquad a(x) = -\frac{1}{x} \quad b(x) = senx \qquad A(x) = -\ln x$

$$y(x) = e^{-\ln x} \cdot \int e^{\ln x} \cdot senx\, dx = \frac{1}{x} \cdot \int \underset{f'}{x} \cdot \underset{g'}{senx}\, dx = \frac{1}{x} \cdot \left[-x\cos x - \int -\cos x\, dx \right]$$

$$= \frac{1}{x} \left[-x \cos x + \underset{+c}{senx} \right] = \frac{senx}{x} - \cos x + \frac{c}{x} \quad c \in R$$

b) $\begin{cases} y' = -\frac{y}{x} + senx \\ y(\pi) = 2 \end{cases} \qquad 2 = \frac{sen\pi}{\pi} - \cos\pi + \frac{c}{\pi} \qquad 2 = +1 + \frac{c}{\pi} \quad c = \pi$

$$y(\cancel{x}) = \frac{senx}{x} - \cos x + \frac{\pi}{\cancel{x}}$$

METODO FATTORE INTEGRANTE

$y'(x) = a(x)\, y(x) + b(x)$

$A(x)$ PRIMITIVA DI $a(x) \rightarrow A(x) = \int a(x)\, dx$

$\underset{\substack{\uparrow \\ \text{FATTORE INTEGRANTE}}}{e^{-A(x)}} \cdot y'(x) = e^{-A(x)} \cdot [a(x) y(x) + b(x)]$

$$e^{-A(x)} \cdot y'(x) - e^{-A(x)} \cdot a(x) \cdot y(x) = e^{-A(x)}\, b(x)$$

$\left(y(x) \cdot e^{-A(x)} \right)' = e^{-A(x)}\, b(x) \qquad$ POI INTEGRO

$$y(x) \cdot e^{-A(x)} = \int e^{-A(x)} \cdot b(x)\, dx \qquad y(x) = e^{A(x)} \int e^{-A(x)} \cdot b(x)\, dx$$

$e^{\ln x} \left(y'(x) + \frac{1}{x} \cdot y \right) = e^{\ln x} senx$

$\left(e^{\ln x} \cdot y(x) \right)' = e^{\ln x} senx$

$\cancel{x} \cdot y(x) = \int \cancel{e^{\ln x}} senx\, dx$

$x\, y(x) = senx - x\cos x + c$

$y(x) = \frac{senx}{x} - \cos x + \frac{c}{x}$

$2 = 0 + 1 + \frac{c}{\pi} \qquad c = \pi$

---

②

a) $y'' + y = \cos x + 2 senx \qquad e^{\alpha x} \cdot \cos x \qquad e^{\alpha x} senx$

1. $\lambda^2 + 1 = 0 \qquad \lambda^2 = -1 \qquad \lambda_1 = i \qquad \lambda_2 = -i \qquad y_0(x) = c_1 \cos x + c_2\, senx \qquad c_1, c_2 \in R$

$\qquad\qquad = \underset{\alpha}{0} + \underset{\beta}{i} \qquad = 0 - i$

2. $v(x) = a\cos x + b\, senx \qquad v'(x) = -a\, senx + b\cos x \qquad v''(x) = -a\cos x - b\, senx$

$-a\cancel{\cos x} - b\cancel{senx} + a\cancel{\cos x} + b\cancel{senx} = \cos x + 2senx \qquad$ perchè sol. $y_0(x)$

$v(x) = ax\cos x + bx\, senx \qquad v'(x) = a\cos x - ax\, senx + b\, senx + bx\cos x \qquad v''(x) = -a\, senx - a\, senx - ax\cos x + b\cos x + b\cos x - bx\, senx$

$$-2a\, senx - \cancel{ax\cos x} + 2b\cos x - \cancel{bx\, senx} + \cancel{ax\cos x} + \cancel{bx\, senx} = \cos x + 2senx$$

$\begin{cases} -2a = 2 \\ 2b = 1 \end{cases} \qquad \begin{cases} a = -1 \\ b = 1/2 \end{cases} \qquad v(x) = -\cos x + \frac{1}{2} senx = y_p(x)$

3. $y_g(x) = C_1\cos x + C_2 \,\text{sen}\, x - x\cos x + x\frac{1}{2}\,\text{sen}\, x$ $\quad y_g'(x) = -C_1 \,\text{sen}\, x + C_2\cos x + x\,\text{sen}\, x + \frac{1}{2}x\cos x - \cos x + \frac{1}{2}\,\text{sen}$

b) $\begin{cases} y'' + y = \cos x + 2\,\text{sen}\, x \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$ $\quad y(0) = 0 \quad y'(0) = 0 \quad \begin{cases} 0 = C_1 - 0 \\ 0 = C_2 - 1 \end{cases} \quad \begin{cases} C_1 = 0 \\ C_2 = 1 \end{cases}$ $\quad y(x) = \text{sen}\, x - x\cos x + \frac{x}{2}\,\text{sen}\, x$

③

a) $y''' + y'' + 4y' + 4y = x + e^{x}$

1. $\lambda^3 + \lambda^2 + 4\lambda + 4 = 0 \quad \lambda^2(\lambda+1) + 4(\lambda+1) = 0 \quad (\lambda+1)(\lambda^2+4) = 0$ $\quad \lambda_1 = -1 \quad y = e^{-x}$

$\lambda_2 = 2i \quad y = \cos 2x$

$\lambda_3 = -2i \quad y = \text{sen}\, 2x$

$y_0(x) = C_1 e^{-x} + C_2\cos 2x + C_3 \,\text{sen}\, 2x$

2. Considero $y''' + y'' + 4y' + 4y = x \quad v(x) = ax + b \quad v'(x) = a \quad v''(x) = v'''(x) = 0$

$4a + 4ax + 4b = x \quad \begin{cases} 4a = 1 \\ 4a + 4b = 0 \end{cases} \quad \begin{cases} a = \frac{1}{4} \\ b = -\frac{1}{4} \end{cases} \quad y_{p_1}(x) = \frac{1}{4}x - \frac{1}{4}$

Considero $y''' + y'' + 4y' + 4y = e^{x} \quad v(x) = ke^{x} \quad v'(x) = ke^{x} \quad v''(x) = ke^{x} \quad v'''(x) = ke^{x}$

$ke^{x} + ke^{x} + 4ke^{x} + 4ke^{x} = e^{x} \quad e^{x}((1+1+4+4)k) = e^{x} \quad k = \frac{1}{10}$

$y_{p_2}(x) = \frac{1}{10}e^{x}$

3. $y_g(x) = C_1 e^{-x} + C_2\cos 2x + C_3\,\text{sen}\, 2x + \frac{1}{4}x - \frac{1}{4} + \frac{1}{10}e^{x}$

b) $y''' - 9y' = x^2 + \cos(3x)$ $\quad$ 1. $\lambda^3 - 9\lambda = 0 \quad \lambda(\lambda^2 - 9) = 0 \quad \lambda_1 = 0 \quad y = 1$

$\lambda_2 = +3 \quad y = e^{3x}$

$y_0(x) = C_1 + C_2 e^{3x} + C_3 e^{-3x}$ $\quad (ax^2 + bx + c)x$ MOLTIPLICO PER X $\quad \lambda_3 = -3 \quad y = e^{-3x}$

2. Considero $y''' - 9y' = x^2$ $\quad$ ~~$v(x) = ax^3 + bx^2 + cx + d$~~ ~~$v'(x) = 3ax^2 + 2bx + c$~~ ~~$v''(x) = 6ax + 2b$~~ ~~$v'''(x) = 6a$~~

~~$6a - (27ax^2 + 18bx + 9c) = x^2$~~ $\quad v(x) = (ax^2 + bx + c)\cdot x$

$v'(x) = 3ax^2 + 2bx + c$

~~$\begin{cases} -27a = 1 \\ -18b = 0 \\ 6a - 9c = 0 \end{cases}$~~ ~~$\begin{cases} a = -\frac{1}{27} \\ b = 0 \\ -\frac{2}{9} - 9c = 0 \end{cases}$~~ ~~$\begin{cases} a = -\frac{1}{27} \\ b = 0 \\ c = -\frac{2}{81} \end{cases}$~~ ~~$y_p(x)$~~ $\quad v''(x) = 6ax + 2b$

$v'''(x) = 6a$

$6a - 27ax^2 - 18bx - 9c = x^2 \quad \begin{cases} -27a = 1 \\ -18b = 0 \\ 6a - 9c = 0 \end{cases} \quad \begin{cases} a = -\frac{1}{27} \\ b = 0 \\ c = -\frac{2}{81} \end{cases} \quad y_{p_1}(x) = -\frac{1}{27}x^3 - \frac{2}{81}x$

Considero $y''' - 9y' = \cos 3x$

$v(x) = a\cos 3x + b\,\text{sen}\, 3x \quad v'(x) = -3a\,\text{sen}\, 3x + 3b\cos 3x \quad v''(x) = -9a\cos 3x - 9b\,\text{sen}\, 3x$

$v'''(x) = 27a\,\text{sen}\, 3x - 27b\cos 3x$ $\quad 27a\,\text{sen}\, 3x - 27b\cos 3x + 27a\,\text{sen}\, 3x - 27b\cos 3x = \cos 3x$

8/5/08

②

$$\begin{cases} 54a = 0 \\ -54b = 1 \end{cases} \quad \begin{cases} a = 0 \\ b = -\frac{1}{54} \end{cases} \quad y_{P_2}(x) = -\frac{1}{54}\,\text{sen}\,3x$$

3. $$y_g(x) = C_1 + C_2 e^{3x} + C_3 e^{-3x} - \frac{1}{27}x^3 - \frac{2}{81}x - \frac{1}{54}\,\text{sen}\,3x$$

④

a) $\begin{cases} y' = x^2y^2 + yx^2 \\ y(0) = 1 \end{cases}$ b) $\begin{cases} y' = x^2y^2 + yx^2 \\ y(0) = 0 \end{cases}$ c) $\begin{cases} y' = x^2y^2 + yx^2 \\ y(0) = -1 \end{cases}$

$$y' = x^2y^2 + yx^2 \quad \frac{dy}{dx} = x^2(y^2+y) \quad \frac{1}{y^2+y}\,dy = x^2dx \quad \int \frac{1}{y^2+y}\,dy = \int x^2dx$$

$y' = x^2y(y+1)$ $\quad y=0$ SOL. $\quad y=-1$ COSTANTI

$$\int \frac{1}{y}\cdot\frac{1}{y+1}\,dy = \frac{x^3}{3} + c \qquad \frac{A}{y} + \frac{B}{y+1} = \frac{1}{y(y+1)}$$

$$A(y+1) + By = 1 \quad \begin{cases} A+B=0 \\ A=1 \end{cases} \quad \begin{cases} A=1 \\ B=-1 \end{cases} \quad \int\frac{dy}{y} + \int -\frac{dy}{y+1} = \frac{x^3}{3} + c$$

$$\ln|y| - \ln|y+1| = \frac{x^3}{3} + c \quad \ln\left|\frac{y}{y+1}\right| = \frac{x^3}{3} + c \quad \frac{y}{y+1} = \pm e^{\frac{x^3}{3}}\cdot e^c$$

$$e^c = C$$

$$\frac{y+1}{y} = e^{-\frac{x^3}{3} - c} \quad 1 + \frac{1}{y} = e^{-\frac{x^3}{3} + c} \quad y = \frac{1}{e^{-\frac{x^3}{3} + c} - 1} = \frac{e^{\frac{1}{3}x^3 + c}}{1 - e^{\frac{1}{3}x^3 + c}} = \frac{Ce^{\frac{1}{3}x^3}}{1 - Ce^{\frac{1}{3}x^3}} \quad C \in \mathbb{R}$$

a) $$1 = \frac{C}{1-C} \quad 1 - C = C \quad C = \frac{1}{2}$$

$$y = \frac{\frac{1}{2}e^{\frac{1}{3}x^3}}{1 - \frac{1}{2}e^{\frac{1}{3}x^3}}$$

b) $$0 = \frac{C}{1-C} \quad C = 0 \quad y = 0$$

c) $$-1 = \frac{C}{1-C} \quad -1 + \cancel{C} = \cancel{C} \quad -1 = 0 \quad C \to +\infty \quad \lim_{C\to+\infty} \frac{Ce^{x^3/3}}{1 - Ce^{x^3/3}} = -1 \quad y = -1$$

ESERCITAZIONE

14/05/08

①

EQUAZIONI DIFF. LINEARI 1° ORDINE
A COEFF. VARIABILI

$$y'(x) = a(x)\,y(x) + b(x)$$

1) $A(x) = \int a(x)\,dx \qquad e^{-A(x)}$ FATTORE INTEGRANTE → MOLTIPLICO TUTTO

2) $y'(x) \cdot e^{-A(x)} = e^{-A(x)} a(x)\,y(x) + e^{-A(x)} b(x)$

$$y'(x)\,e^{-A(x)} - e^{-A(x)} a(x)\,y(x) = e^{-A(x)} b(x)$$

$$\left(y(x) \cdot e^{-A(x)}\right)' = e^{-A(x)} b(x)$$ INTEGRO

$$y(x) \cdot e^{-A(x)} = \int e^{-A(x)} b(x)\,dx + c \qquad y(x) = e^{A(x)} \cdot c + e^{A(x)} \int e^{-A(x)} b(x)\,dx$$

ES. 4.5.1.

- $y' = 3y - \cos x \qquad \underbrace{y' - 3y}_{} = -\cos x$

① OMOGENEA ASSOCIATA
② SOL. PARTICOLARE

QUESTA È A COEFF. COSTANTI

$$\lambda - 3 = 0 \quad \lambda = 3 \qquad y_0(x) = c e^{3x}$$

$$f(x) = -\cos x \qquad v(x) = a \operatorname{sen} x + b \cos x \qquad v'(x) = a \cos x - b \operatorname{sen} x$$

$$a \cos x - b \operatorname{sen} x - 3(a \operatorname{sen} x + b \cos x) = -\cos x \qquad \operatorname{sen} x(-b - 3a) + \cos x(a - 3b) = -\cos x$$

$$\begin{cases} -b - 3a = 0 \\ a - 3b = -1 \end{cases} \quad \begin{cases} b = -3a \\ a + 9a = -1 \end{cases} \quad \begin{cases} a = -\frac{1}{10} \\ b = +\frac{3}{10} \end{cases} \qquad y_p(x) = -\frac{1}{10} \operatorname{sen} x + \frac{3}{10} \cos x$$

$$y(x) = y_0(x) + y_p(x) = c e^{3x} - \frac{1}{10} \operatorname{sen} x + \frac{3}{10} \cos x.$$

- $y' = \dfrac{y}{2x} + \log x \qquad D: \underline{x > 0} \qquad a(x) = \dfrac{1}{2x} \qquad A(x) = \displaystyle\int \frac{1}{2x}\,dx = \frac{1}{2} \int \frac{1}{x}\,dx = \frac{1}{2} \log|x| \underset{x>0}{\downarrow} = \frac{1}{2} \log x$

$b(x) = \log x$

FATT. INTEG.

$$e^{-\frac{1}{2}\log x} = e^{\log x^{-\frac{1}{2}}} = e^{\log \frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}}$$

$$y' \cdot \frac{1}{\sqrt{x}} = \frac{y}{2x} \cdot \frac{1}{\sqrt{x}} + \frac{\log x}{\sqrt{x}} \qquad \left(y \cdot \frac{1}{\sqrt{x}}\right)' = \frac{\log x}{\sqrt{x}} \qquad y \cdot \frac{1}{\sqrt{x}} = \int \frac{\log x}{\sqrt{x}}\,dx + c$$

$$y = \sqrt{x} \int \frac{\log x}{\sqrt{x}}\,dx + c \cdot \sqrt{x}$$ RISOLVO L'INTEGRALE

$$\int \underset{f}{\log x} \; \underset{g'}{\frac{1}{\sqrt{x}}} \, dx = \text{[illegible]}$$

$$u = \log x \longrightarrow du = \frac{1}{x}dx$$
$$dv = x^{-\frac{1}{2}}dx \longrightarrow v = 2\sqrt{x}$$

$$= 2\sqrt{x}\log x - 2\int \frac{1}{x}\sqrt{x}\,dx = 2\sqrt{x}\log x - 2 \cdot 2\sqrt{x}$$

$$y(x) = \sqrt{x}\left(2\sqrt{x}\log x - 4\sqrt{x}\right) + c\sqrt{x} = 2x\log x - 4x + c\sqrt{x}$$

- $x^2 y' = -2y - 3 \qquad x \neq 0$

$$y' = -\frac{2}{x^2}y - \frac{3}{x^2} \qquad a(x) = -\frac{2}{x^2} \qquad A(x) = \int -\frac{2}{x^2}dx = 2 \cdot \frac{1}{x} = \frac{2}{x} \qquad \text{FAT. INT. } e^{-A(x)} = e^{-\frac{2}{x}}$$

$$y' \cdot e^{-\frac{2}{x}} = -\frac{2}{x^2}y \cdot e^{-\frac{2}{x}} - \frac{3}{x^2} \cdot e^{-\frac{2}{x}} \qquad y'e^{-\frac{2}{x}} + \frac{2}{x^2}ye^{-\frac{2}{x}} = -\frac{3}{x^2}e^{-\frac{2}{x}}$$

$$\left(ye^{-\frac{2}{x}}\right)' = -\frac{3}{x^2}e^{-\frac{2}{x}} \qquad ye^{-\frac{2}{x}} = \int -\frac{3}{x^2}e^{-\frac{2}{x}}dx + c$$

$$y = e^{\frac{2}{x}}\int -\frac{3}{x}e^{-\frac{2}{x}}dx + e^{\frac{2}{x}} \cdot c \qquad \int -\frac{3}{x^2}e^{-\frac{2}{x}}dx = \frac{-3}{2}e^{-\frac{2}{x}}$$

$$\left(e^{-\frac{2}{x}}\right)' = e^{-\frac{2}{x}} \cdot (-2) \cdot \left(-\frac{1}{x^2}\right)$$

$$y = e^{\frac{2}{x}}\left(-\frac{3}{2}\right)e^{-\frac{2}{x}} + e^{\frac{2}{x}} \cdot c = -\frac{3}{2} + e^{\frac{2}{x}} \cdot c$$

$$y' = e^{-2x} - \frac{y}{x} \quad \underline{x \neq 0} \qquad a(x) = -\frac{1}{x} \qquad A(x) = \int -\frac{1}{x}dx = -\ln|x| = \ln\left|\frac{1}{x}\right| \qquad \text{F.I. } e^{+\ln|x|} = |x|$$

$$y' \cdot |x| = |x|e^{-2x} - \cancel{|x|} \cdot \frac{y}{\cancel{x}} \qquad \text{se } x>0 \quad y'x + y = xe^{-2x} \qquad (yx)' = xe^{-2x}$$

se $x<0$
$$-y'x = -xe^{-2x} + x\frac{y}{x} \quad \text{UGUAL A PRI}$$

$$yx = \int xe^{-2x}dx + c \qquad y = \frac{1}{x}\int xe^{-2x}dx + \frac{c}{x} \qquad \int \underset{f}{x}\underset{g'}{e^{-2x}}dx = -\frac{1}{2}e^{-2x} \cdot x - \int -\frac{1}{2}e^{-2x} \cdot 1\,dx$$

$$u = x \rightarrow du = 1$$
$$dv = e^{-2x} \cdot dx \rightarrow -\frac{1}{2}e^{-2x} = v$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\frac{1}{-2}\int -2e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

$$y = \frac{1}{x}\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) + \frac{c}{x} \qquad y = -\frac{1}{2}e^{-2x} - \frac{1}{4x}e^{-2x} + \frac{c}{x}$$

$y' = y \log x + x^x$ $\quad a(x) = \log x \quad A(x) = \int \log x\,dx = \int \underset{g'}{1} \cdot \underset{f}{\log x} = x \log x - \int x \cdot \frac{1}{x}\,dx = x\log x - x$

$x > 0 \quad$ F.I. $= e^{-x\log x + x} \left[= e^{x} \cdot e^{-\log x^{x}} = e^{x} \cdot \frac{1}{x^{x}} \right]$

$$y' \cdot e^{-x\log x + x} = y \log x \cdot e^{-x\log x + x} + x^{x} e^{-x\log x + x}$$

$$\left(y \cdot e^{-x\log x + x}\right)' = \cancel{x^{x}} \cdot e^{x} \cdot \frac{1}{\cancel{x^{x}}} \qquad y \cdot e^{-x\log x + x} = \int e^{x}\,dx + c$$

$$y = \frac{e^{x}}{e^{-x\log x + x}} + \frac{c}{e^{-x\log x + x}} \qquad y = e^{x\log x} + \frac{c}{e^{-x\log x + x}} = x^{x} + \frac{c}{e^{-x\log x + x}} = x^{x} + \frac{c x^{x}}{e^{x}}$$

- $y' = y\cos x + \operatorname{sen}x \cos x$ $\quad a(x) = \cos x \quad A(x) = \operatorname{sen} x \quad$ F.I. $e^{-\operatorname{sen}x}$

$$y' \cdot e^{-\operatorname{sen}x} = y \cdot \cos x \cdot e^{-\operatorname{sen}x} + \operatorname{sen}x \cos x\, e^{-\operatorname{sen}x}$$

$$y' \cdot e^{-\operatorname{sen}x} - y\cos x\, e^{-\operatorname{sen}x} = \operatorname{sen}x\cos x\, e^{-\operatorname{sen}x} \qquad \left(y\, e^{-\operatorname{sen}x}\right)' = \operatorname{sen}x\cos x\, e^{-\operatorname{sen}x}$$

$$y\, e^{-\operatorname{sen}x} = \int \operatorname{sen}x\cos x\, e^{-\operatorname{sen}x}\,dx + c \qquad y = e^{\operatorname{sen}x}\int \operatorname{sen}x\cos x\, e^{-\operatorname{sen}x}\,dx + c\, e^{\operatorname{sen}x}$$

$$\int \underset{f}{\operatorname{sen}x} \cdot \underset{g'}{\cos x\, e^{-\operatorname{sen}x}}\,dx = -\operatorname{sen}x\, e^{-\operatorname{sen}x} - \int -\cos x\, e^{-\operatorname{sen}x}\,dx = -\operatorname{sen}x\, e^{-\operatorname{sen}x} - e^{-\operatorname{sen}x}$$

$$y = e^{\operatorname{sen}x}\left(-\operatorname{sen}x\, e^{-\operatorname{sen}x} - e^{-\operatorname{sen}x}\right) + c\, e^{\operatorname{sen}x}$$

$$y = -\operatorname{sen}x - 1 + c\, e^{\operatorname{sen}x}$$

La soluzione della eq. diff. $\begin{cases} y''(x) - y(x) = e^{2x} \\ y(0) = \frac{1}{3} \\ y'(0) = 1 \end{cases}$ è

[A] $\frac{2e^{x}}{3} - \frac{e^{-x}}{3}$ $\qquad$ [C] $-\frac{e^{x}}{6} + \frac{e^{-x}}{6} + \frac{e^{2x}}{3}$

[✓ B] $\frac{e^{x}}{6} - \frac{e^{-x}}{6} + \frac{e^{2x}}{3}$ $\qquad$ [D] $\frac{e^{2x}}{3}$

OMOGENEA

$$y'' - y = 0 \quad \lambda^2 - 1 = 0 \quad \lambda = \pm 1$$

$$y_0(x) = c_1 e^{x} + c_2 e^{-x}$$

PARTICOLARE

$v(x) = k e^{2x}$ va bene perchè non coincide con $\quad v'(x) = 2k e^{2x} \quad v''(x) = 4k e^{2x}$

$$4k e^{2x} - k e^{2x} = e^{2x} \qquad 3k e^{2x} = e^{2x} \qquad 3k = 1 \qquad k = \frac{1}{3}$$

$y_p(x) = \frac{1}{3} e^{2x} \qquad y_G(x) = \frac{e^{2x}}{3} + c_1 e^{x} + c_2 e^{-x}$ $\quad$ LA [A] NON PUÒ ESSERE

$$\frac{1}{3} = \frac{e^{0}}{3} + c_1 e^{0} + c_2 e^{0} \qquad c1 + c2 = 0 \qquad y_G'(x) = \frac{2}{3} e^{2x} + c_1 e^{x} - c_2 e^{-x}$$

$y_G'(0) = \frac{2}{3} + c_1 - c_2 = 1$ $\begin{cases} c_1 - c_2 = \frac{1}{3} \\ c_1 = -c_2 \end{cases}$ $\begin{cases} c_1 = -c_2 \\ -c_2 - c_2 = \frac{1}{3} \end{cases}$ $\begin{cases} c_2 = -\frac{7}{6} \\ c_1 = \frac{1}{6} \end{cases}$

$y_g(x) = \frac{1}{6} e^{x} - \frac{1}{6} e^{-x} + \frac{e^{2x}}{3}$ è la B.

Per quali valori di $a, b, c \in \mathbb{R}$ la funzione $e^{2x} + e^{-x} + 1$ è soluzione della equazione $y''' + ay'' + by' + cy = 0$

[A] $a=0$, $b=-4$, $c=0$ [C] $a=1$, $b=-2$, $c=$ qualsiasi

[✓ B] $a=-1$, $b=-2$, $c=0$ [D] $a=1$, $b=0$, $c=-2$

~~$\lambda^3 + a\lambda^2 + b\lambda + c = 0$~~ $y = e^{2x} + e^{-x} + 1$ $y'' = 4e^{2x} + e^{-x}$

$y' = 2e^{2x} - e^{-x}$ $y''' = 8e^{2x} - e^{-x}$

$8e^{2x} - e^{-x} + a(4e^{2x} + e^{-x}) + b(2e^{2x} - e^{-x}) + c(e^{2x} + e^{-x} + 1) = 0$

$8e^{2x} - e^{-x} + 4ae^{2x} + ae^{-x} + 2be^{2x} - be^{-x} + ce^{2x} + ce^{-x} + c = 0$

$\begin{cases} 8 + 4a + 2b + c = 0 \\ -1 + a - b + c = 0 \\ c = 0 \end{cases}$ $\begin{cases} c = 0 \\ a = 1 + b \\ 8 + 4 + 4b + 2b = 0 \end{cases}$ $\begin{cases} c = 0 \\ b = -2 \\ a = -1 \end{cases}$

$y'' - 3y' - 10y = 0$ è verificata da

[A] nessuna delle altre risposte è vera [✓ B] e^{5x}

[C] $c_1 e^{5x} + c_2 e^{-2x} + 1$ $c_1, c_2 \in \mathbb{R}$ [D] $c_1 e^{-5x} + c_2 e^{2x}$

$\lambda^2 - 3\lambda - 10 = 0$ $(\lambda - 5)(\lambda + 2) = 0$ $\lambda = 5$ $\lambda = -2$ $y = c_1 e^{5x} + c_2 e^{-2x}$

$\begin{cases} y' = (1+y^2)\,\text{sen}x \\ y(0) = 1 \end{cases}$ $\frac{y'}{1+y^2} = \text{sen}x$ $\int \frac{1}{1+y^2}\,dy = \int \text{sen}x\,dx$

$\text{arctg}\, y = -\cos x + c$ $y = \text{tg}(-\cos x + c)$ $y(0) = 1$

$1 = \text{tg}(-1 + c)$ $c - 1 = \text{arctg}\, 1$ $c - 1 = \frac{\pi}{4}$ $c = \frac{1}{4}\pi + 1$

$y = \text{tg}\left(-\cos x + 1 + \frac{1}{4}\pi\right)$

ESERCITAZIONE

15/05/0

①

4.5.6.

i) $y'' + y' = x$ $\quad y'' + y' = 0 \quad \lambda^2 + \lambda = 0 \quad \lambda = 0,\ \lambda = -1 \quad y_0(x) = c_1 + c_2 e^{-x}$

$v(x) = ax + b$ ma b è già parte della soluzione $\Rightarrow v(x) = (ax+b)x = ax^2 + bx$

$v(x)' = 2ax + b \quad v''(x) = 2a \quad 2a + 2ax + b = x \quad \begin{cases} 2a + b = 0 \\ 2a = 1 \end{cases} \quad \begin{cases} a = 1/2 \\ b = -1 \end{cases}$

$y_p = \frac{1}{2}x^2 - x \qquad y_g(x) = c_1 + c_2 e^{-x} + \frac{1}{2}x^2 - x$

ii) $y'' + y' = xe^{-x}$ $\quad y'' + y' = 0 \quad \lambda^2 + \lambda = 0 \quad \lambda = 0,\ \lambda = -1 \quad y_0(x) = c_1 + c_2 e^{-x}$

$f(x) = x \cdot e^{-x} \quad v(x) = (ax+b) \cdot e^{-x}$ (il K è incluso nelle costanti a e b)

$= axe^{-x} + \underbrace{be^{-x}}$ → fa parte di $y_0(x)$

$v(x) = ax^2e^{-x} + bxe^{-x} \qquad v'(x) = 2axe^{-x} - ax^2e^{-x} + be^{-x} - bxe^{-x}$

$v''(x) = 2ae^{-x} - 2axe^{-x} - 2axe^{-x} + ax^2e^{-x} - be^{-x} - be^{-x} + bxe^{-x}$

$2ae^{-x} - 4axe^{-x} \cancel{+ax^2e^{-x}} - 2be^{-x} \cancel{+bxe^{-x}} + 2axe^{-x} \cancel{-ax^2e^{-x}} + be^{-x} \cancel{-bxe^{-x}} = xe^{-x}$

$e^{-x}(2a - 2ax - b) = e^{x}(x) \qquad \begin{cases} 2a - b = 0 \\ -2a = 1 \end{cases} \quad \begin{cases} a = -1/2 \\ b = -1 \end{cases}$

$y_p(x) = -\frac{1}{2}x^2e^{-x} - xe^{-x} \qquad y_g(x) = c_1 + c_2e^{-x} - \frac{1}{2}x^2e^{-x} - xe^{-x}$

iii) $y''' - 2y'' = x^2 + e^{2x} \quad \lambda^3 - 2\lambda^2 = 0 \quad \lambda^2(\lambda - 2) = 0 \quad \lambda = 0$ DOPPIA $\quad \lambda = 2 \quad y_0(x) = c_1 + c_2x + c_3e^{2x}$

$f_1(x) = x^2 \quad v_1(x) = ax^2 + \underbrace{bx + c}$ → già contenute

$v_1(x) = ax^4 + bx^3 + cx^2 \qquad v_1'(x) = 4ax^3 + 3bx^2 + 2cx \quad v_1''(x) = 12ax^2 + 6bx + 2c$

$v_1'''(x) = 24ax + 6b \qquad 24ax + 6b - 24ax^2 - 12bx - 4c = x^2 \quad \begin{cases} -24a = 1 \\ 24a - 12b = 0 \\ 6b - 4c = 0 \end{cases} \quad \begin{cases} a = -1/24 \\ b = -1/12 \\ c = -1/8 \end{cases}$

$y_{p_1}(x) = -\frac{1}{24}x^4 - \frac{1}{12}x^3 - \frac{1}{8}x^2$

$v_2(x) = ke^{2x}$ già contenuta $\quad v_2(x) = kxe^{2x} \quad v_2'(x) = ke^{2x} + 2kxe^{2x}$

$v_2''(x) = 2ke^{2x} + 2ke^{2x} + 4kxe^{2x} = 4ke^{2x} + 4kxe^{2x} \quad v_2'''(x) = 8ke^{2x} + 8kxe^{2x} + 4ke^{2x} = 12ke^{2x} + 8kxe^{2x}$

$12ke^{2x} \cancel{+8kxe^{2x}} - 8ke^{2x} \cancel{-8kxe^{2x}} = e^{2x} \qquad e^{2x}(4k) = e^{2x} \quad 4k = 1 \quad k = \frac{1}{4} \quad y_{p_2}(x) = \frac{1}{4}xe^{2x}$

$$y_g(x) = c_1 + c_2 x + c_3 e^{2x} - \frac{x^4}{24} - \frac{x^3}{12} - \frac{x^2}{8} + \frac{1}{4} x e^{2x}$$

$$\boxed{\begin{array}{l} \text{se } \lambda = \alpha \pm i\beta \\ y(x) = c_1 e^{\alpha x} \operatorname{sen}\beta x + c_2 e^{\alpha x} \cos\beta x \end{array}}$$

IV) $y'' + y = \cos x \qquad \lambda^2 + 1 = 0 \qquad \lambda^2 = -1 \qquad \lambda = \pm i \quad \begin{smallmatrix} \alpha = 0 \\ \beta = 1 \end{smallmatrix}$

$y_0(x) = c_1 \operatorname{sen} x + c_2 \cos x$

$v(x) = (a \operatorname{sen} x + b \cos x)\, x = a x \operatorname{sen} x + b x \cos x \qquad v'(x) = a \operatorname{sen} x + a x \cos x + b \cos x - b x \operatorname{sen} x$

$v''(x) = a \cos x + a \cos x - a x \operatorname{sen} x - b \operatorname{sen} x - b \operatorname{sen} x - b x \cos x = 2a \cos x - b x \cos x - a x \operatorname{sen} x - 2b \operatorname{sen} x$

$2a \cos x \cancel{- b x \cos x} \cancel{- a x} \operatorname{sen} x - 2b \operatorname{sen} x + \cancel{a x \operatorname{sen} x} + \cancel{b x \cos x} = \cos x \qquad \begin{cases} 2a = 1 \\ -2b = 0 \end{cases} \quad \begin{cases} a = 1/2 \\ b = 0 \end{cases}$

$y_p(x) = \frac{1}{2} x \operatorname{sen} x \qquad y_g(x) = c_1 \operatorname{sen} x + c_2 \cos x + \frac{1}{2} x \operatorname{sen} x$

V) $y''' - 5y'' + 9y' - 5y = x + e^{2x}$

$\lambda^3 - 5\lambda^2 + 9\lambda - 5 = 0$ si annulla per $\lambda = 1$ (divisori termine noto)

$$\begin{array}{c|ccc|c} & 1 & -5 & 9 & -5 \\ 1 & & 1 & -4 & 5 \\ \hline & 1 & -4 & 5 & 0 \end{array} \qquad (\lambda^2 - 4\lambda + 5)(\lambda - 1).$$

$(\lambda - 1)(\lambda^2 - 4\lambda + 5) = 0$

$\lambda = 1$

$\lambda^2 - 4\lambda + 5 = 0 \quad \lambda = \frac{2 \pm \sqrt{4-5}}{1} = 2 \pm i \qquad y_0(x) = c_1 e^x + c_2 e^{2x} \operatorname{sen} x + c_3 e^{2x} \cos x$

$v_1(x) = ax + b \qquad v_1'(x) = a \quad v_1''(x) = 0 \quad v_1'''(x) = 0 \qquad 9a - 5ax - 5b = x \quad \begin{cases} -5a = 1 \\ 9a - 5b = 0 \end{cases} \quad \begin{cases} a = -\frac{1}{5} \\ -5b = \frac{9}{5} \quad b = -\frac{9}{25} \end{cases}$

$y_{1p}(x) = -\frac{1}{5} x - \frac{9}{25}$

$v_2(x) = k e^{2x} \qquad v_2'(x) = 2k e^{2x} \qquad v_2''(x) = 4k e^{2x} \qquad v_2'''(x) = 8k e^{2x}$

$8k e^{2x} - 20k e^{2x} + 18k e^{2x} - 5k e^{2x} = e^{2x} \qquad k = 1 \qquad y_{p2}(x) = e^{2x}$

$y_g(x) = c_1 e^x + c_2 e^{2x} \operatorname{sen} x + c_3 e^{2x} \cos x - \frac{1}{5} x - \frac{9}{25} + e^{2x}$

QUESITI

L'equazione differenziale $y'(x) + 3x^2 y(x) = -x^2$ è verificata da

[A] $c e^{x^3} - \frac{1}{3}$ $c \in \mathbb{R}$ [C] $c e^{-x^3}$ $c \in \mathbb{R}$

[✓ B] $2 e^{-x^3} - \frac{1}{3}$ aa [D] $c e^{-x^3} + \frac{1}{3}$ $c \in \mathbb{R}$

$y'(x) = -3x^2 y(x) - x^2 \qquad a(x) = -3x^2 \qquad A(x) = \int -3x^2\, dx = -x^3 \qquad \text{F.I} = e^{x^3}$

$y'(x) \cdot e^{x^3} = -3x^2 y(x) \cdot e^{x^3} - e^{x^3} \cdot x^2 \qquad (y e^{x^3})' = -e^{x^3} \cdot x^2 \qquad y e^{x^3} = -\frac{1}{3} \int 3x^2 e^{x^3}\, dx + c$

$y = \frac{1}{e^{x^3}} \cdot \left(-\frac{1}{3} e^{x^3}\right) + \frac{c}{e^{x^3}} \qquad y = \frac{c}{e^{x^3}} - \frac{1}{3}$

L'eq. diff. $y''(x)+y'(x)-2y(x)=2$ è verificata da 15/05/8

②

[A] $c_1e^{-x}+c_2e^{2x}-1$ $\lambda^2+\lambda-2=0$ $(\lambda+2)(\lambda-1)=0$ $\lambda=+1$

[B] $3e^{-2x}$ $\lambda=-2$ $y_0(x)=C_1e^{-2x}+C$

$v(x)=K$ $v'(x)=0\dots$

☒ e^x-1 $\leftarrow \begin{matrix} c_2=1 \\ c_1=0 \end{matrix}$ $-2K=2$ $K=-1$ $y_g=c_1e^{-2x}+c_2e^{x}-1$

[D] $c_1e^x+c_2e^{-2x}$

4.5.5.

$y'''=0$ $\lambda^3=0$ $\underset{\text{TRIPLA}}{\lambda=0}$ $y=c_1e^{0x}+c_2e^{0x}\cdot x+c_3e^{0x}\cdot x^2=c_1+c_2x+x^2c_3$

$y=c_1+c_2x+c_3x^2$

$y'''-4y''+6y'-4y=0$ $\lambda^3-4\lambda^2+6\lambda-4=0$ si annulla per $\lambda=2$

$$\begin{array}{c|ccc|c} & 1 & -4 & 6 & -4 \\ 2 & & 2 & -4 & 4 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$(\lambda-2)(\lambda^2-2\lambda+2)=0$

$\lambda=2$

$\lambda^2-2\lambda+2=0$ $\lambda=\dfrac{1\pm\sqrt{1-2}}{1}=1\pm i$ $y(x)=c_1e^{2x}+c_2e^x\operatorname{sen}x+c_3e^x\cos x$

$y^{IV}+2y''+y=0$ $\lambda^4+2\lambda^2+1=0$ $(\lambda^2+1)^2=0$ $\lambda^2=-1$ $\lambda=\pm i$ SOLUZIONI DOPPIE $\lambda=+i$ DOP $\lambda=-i$ DOPP

$y=c_1\operatorname{sen}x+c_2x\operatorname{sen}x+c_3\cos x+c_4x\cos x$

5.3.7.

ESERCITAZIONE

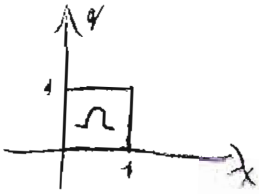
21/05/08

①

$$\int_\Omega f(xy)\,dx\,dy$$

i) $f(xy) = xy(x+y)$

$\Omega = [0,1]\times[0,1]$



NORM. RISPETTO A X

$$\int_0^1 dx \int_0^1 xy(x+y)\,dy = \int_0^1 dx \left[x\int_0^1 (xy+y^2)dy\right] = \int_0^1 dx \left(x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_0^1\right) =$$

$$= \int_0^1 dx \cdot \left(\frac{x^2}{2}+\frac{x}{3}\right) \cancel{dx} = \left[\frac{1}{2}\frac{x^3}{3} + \frac{1}{3}\cdot\frac{x^2}{2}\right]_0^1 = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

ii) $f(xy) = \dfrac{y}{1+xy}$ $\quad\Omega = [0,1]\times[0,1]$

NORM RISPETTO A X

$$\int_0^1 dx \int_0^1 \frac{y}{1+xy}\,dy = \ldots \text{divisione tra polinomi} \ldots$$

NORM RISP. A Y

$$\int_0^1 dy \int_0^1 \frac{y}{1+xy}\,dx = \int_0^1 dy \left[\cancel{y}\ln|1+yx|\right]_0^1 = \int_0^1 dy \cdot \left(\ln|1+y| - \cancel{\ln 1}\right) =$$

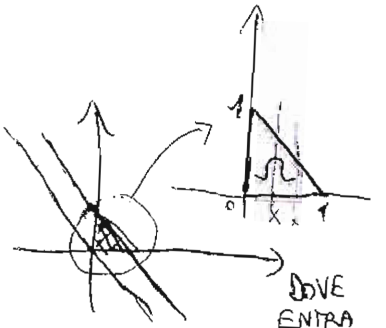
$y+1>0$ per y da 0 a 1

$$= \int_0^1 1\cdot\ln(1+y)\,dy = \int_0^1 \underset{f'}{1}\cdot \underset{g}{\ln(1+y)}\,dy = \left[y\ln(1+y) - \int y\cdot\frac{1}{1+y}dy\right]_0^1 =$$

$$= \left[y\ln(1+y) - \int \frac{1+y-1}{1+y}dy\right]_0^1 = \left[y\ln(1+y) - \left(y - \ln(1+y)\right)\right]_0^1 =$$

$$= \left[(1+y)\ln(1+y) - y\right]_0^1 = 2\ln 2 - 1$$

iii) $f(xy) = \operatorname{sen}(x+y)$ $\quad \Omega = \{(x,y)\in\mathbb{R}^2 : 0\le x+y\le 1,\ 0\le x,\ 0\le y\}$



$$\begin{cases} y \geq -x \\ y \leq 1-x \\ x \geq 0 \\ y \geq 0 \end{cases}$$

DOVE ENTRA RETTA ($y=0$) E DOVE ES ($y=-x+1$)

NORM. RISP. A X

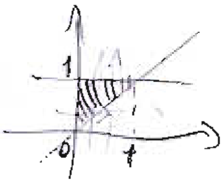
$$\int_0^1 dx \int_0^{-x+1} \operatorname{sen}(x+y)\,dy = \int_0^1 dx \left[-\cos(x+y)\right]_0^{-x+1} =$$

$$= \int_0^1 \left[-\cos(x \cancel{x}+1) - \left(-\cos(x+0)\right)\right]dx = \int_0^1 \left[-\cos 1 + \cos x\right]dx =$$

$$= \left[-x\cos 1 + \operatorname{sen} x\right]_0^1 = -\cos 1 + \operatorname{sen} 1$$

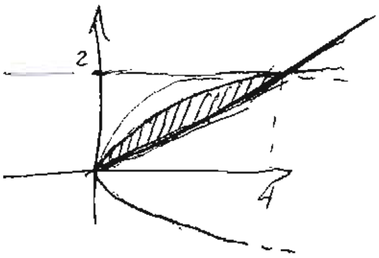
5.3.8.

$$i)\ \int_0^1 dy \int_0^y f(x,y)\,dx \qquad S = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le y\}$$

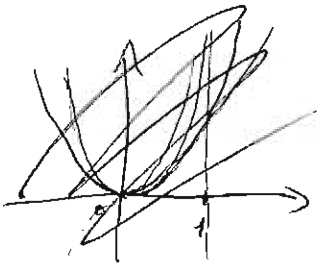


$$\int_0^1 dx \int_x^1 f(x,y)\,dy$$

$$ii)\ \int_0^2 dy \left(\int_{y^2}^{2y} f(x,y)\,dx \right) \qquad S = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2,\ y^2 \le x \le 2y\}$$

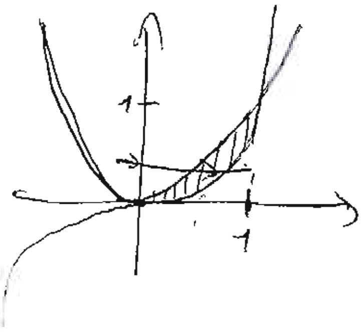


$$\int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x,y)\,dy$$



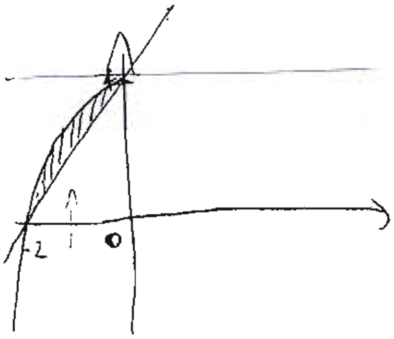
$$iii)\ \int_0^1 dx \int_{x^3}^{x^2} f(x,y)\,dy \qquad S = \{(x,y) \in \mathbb{R}^2 ;\ 0 \le x \le 1,\ x^3 \le y \le x^2\}$$

$$\int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)\,dx$$



$$iv)\ \int_0^4 dy \int_{-\sqrt{4-y}}^{\frac{y-4}{2}} dx\, f(x,y) \qquad S = \{(x,y) \in \mathbb{R}^2 ;\ 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \frac{y-4}{2}\}$$

$$\int_{-2}^0 dx \int_{2x+4}^{-x^2+4} dy\, f(x,y)$$



$x = -\sqrt{4-y} \qquad x^2 = 4 - y$

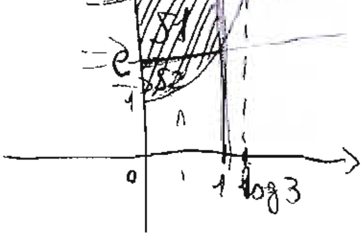
$$\begin{cases} y = -x^2+4 \\ x \le 0 \end{cases}$$

$x = \frac{y}{2} - 2 \qquad y = 2x+4$

$$v)\ \int_0^1 dx \int_{e^x}^3 dy\, f(x,y) \qquad S = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ e^x \le y \le 3\}$$

~~$\int_1^3 dy \int_0$~~

$$\int_{S_1} f(x,y)\,dx\,dy + \int_{S_2} f(x,y)\,dx\,dy = \int_e^3 dy \int_0^1 dx\, f(x,y) + \int_1^e dy \int_0^{\log y} dx\, f(x,y)$$



SPEZZO

$$\begin{cases} y = e^x \\ x = 1 \end{cases} \qquad y = e$$

T.E. 15/01/08

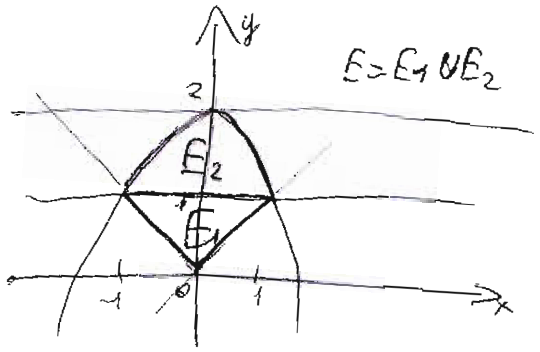
$f(x,y) = y \qquad I = \int_0^1 dy \int_{-y}^{y} f(x,y)\,dx + \int_1^2 dy \int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\,dx$

Disegnare E tale che 21/05/08
$I = \int_E f(x,y)\,dx\,dy$ ②

Invertire l'ordine d'integrazione

$E_1 : \{(x,y) \in \mathbb{R}^2 .\ 0 \le y \le 1 ,\ -y \le x \le y\}$

$E_2 = \{(x,y) \in \mathbb{R}^2 : 1 \le y \le 2 ,\ -\sqrt{2-y} \le x \le \sqrt{2-y}\}$



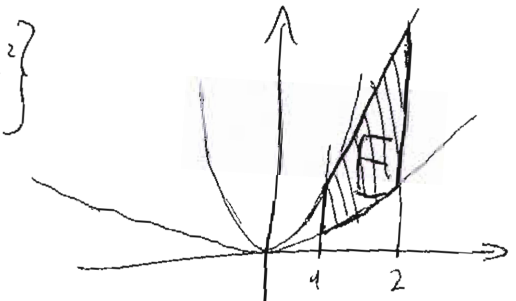
$E = E_1 \cup E_2$

$x = -\sqrt{2-y} \qquad x^2 = 2-y \qquad y = 2-x^2$

$$\int_{-1}^{0} dx \int_{-x}^{-x^2+2} dy\ f(x,y) + \int_0^1 dx \int_x^{-x^2+2} dy\ f(x,y)$$

5.3.9 - Calcolare

$\int_E \frac{x}{x^2+y^2}\,dx\,dy \qquad E = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 2 ,\ \frac{x^2}{2} \le y \le x^2\}$



NOR.
RISP.
x

$$\int_1^2 dx \int_{\frac{x^2}{2}}^{x^2} dy \frac{x}{x^2+y^2} = \int_1^2 dx \cdot \int_{\frac{x^2}{2}}^{x^2} \frac{\cancel{x}}{x^{\cancel{2}}\left(1+\left(\frac{y}{x}\right)^2\right)} dy =$$

$$= \int_1^2 dx \int_{\frac{x^2}{2}}^{x^2} \frac{1}{x} \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} dy = \int_1^2 \left(\mathrm{arctg}\,\frac{x^{\cancel{2}}}{\cancel{x}} - \mathrm{arctg}\,\frac{\frac{x^{\cancel{2}}}{2}}{\cancel{x}}\right) dx = \int_1^2 \left(\mathrm{arctg}\,x - \mathrm{arctg}\,\frac{x}{2}\right) dx$$

$$\int \mathrm{arctg}\,x\,dx = x\,\mathrm{arctg}\,x - \int x \cdot \frac{1}{1+x^2}\,dx = x\,\mathrm{arctg}\,x - \frac{1}{2}\log(1+x^2)$$

$$\int \mathrm{arctg}\,\frac{x}{2}\,dx = x\,\mathrm{arctg}\,\frac{x}{2} - \int x \cdot \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx = x\,\mathrm{arctg}\,\frac{x}{2} - \cancel{\log\left(1+\left(\frac{x}{2}\right)^2\right)} - \log(4+x^2)$$

$$\frac{1}{\cancel{\frac{4+x^2}{4}}} = \frac{4\,2}{4+x^2} \cdot \frac{1}{2}$$

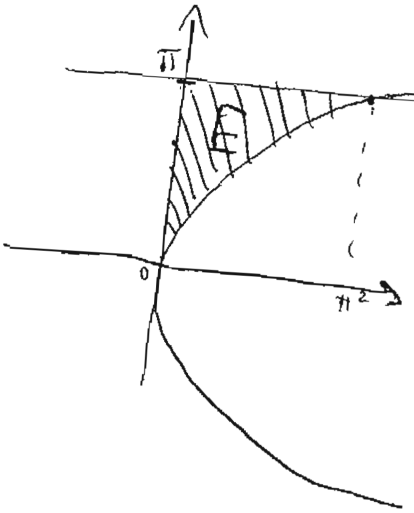
$$\left| x\,\mathrm{arctg}\,x - \frac{1}{2}\log(1+x^2) - x\,\mathrm{arctg}\,\frac{x}{2} + \log(4+x^2) \right|_1^2 = 2\,\mathrm{arctg}\,2 - \frac{1}{2}\log 5 - 2 \cdot \frac{\pi}{4} + \log 8 - \frac{\pi}{4} + \frac{1}{2}\log 2 + \mathrm{arctg} - \log 5$$

= . . .

5.3.10

$$\int_E \frac{(\operatorname{sen} y)^2}{y}\,dx\,dy \qquad y\neq 0 \qquad E=\left\{(x,y)\in\mathbb{R}^2: 0\le y\le\pi,\ 0\le x\le y^2\right\}$$

$$\begin{cases} y=\pi \\ x=y^2 \end{cases} \qquad x=\pi^2$$



DOM. NORM. X

$$\int_0^{\pi^2} dx \int_{\sqrt{x}}^{\pi} \frac{(\operatorname{sen} y)^2}{y}\,dy =$$ NON INTEGRABILE CON FUNZIONI ELEMENTARI

DOM. NORM. Y

$$\int_0^{\pi} dy \int_0^{y^2} \frac{(\operatorname{sen} y)^2}{y}\,dx = \int_0^{\pi} dy \cdot \frac{(\operatorname{sen} y)^2}{\cancel{y}} \cdot y^{\cancel{2}} = \int_0^{\pi} \underset{g}{y} \cdot \underset{f}{(\operatorname{sen} y)^2}\,dy \qquad \operatorname{sen}^2 y = \frac{1-\cos 2y}{2}$$

$$= \int_0^{\pi} y \cdot (1-\cos 2y) \cdot \frac{1}{2}\,dy = \frac{1}{2}\int (y - y\cos 2y)\,dy = \frac{1}{2}\cdot\frac{y^2}{2} - \frac{1}{2}\int \underset{g}{y}\,\underset{f'}{\cos 2y}\,dy =$$

$$= \left[\frac{y^2}{4} - \frac{1}{2}\left(y \cdot \frac{\operatorname{sen} 2y}{2} - \int \frac{\operatorname{sen} 2y}{2}\,dy \right) = \frac{y^2}{4} - \frac{1}{2}\left[y \cdot \frac{\operatorname{sen} 2y}{2} - \frac{1}{4}(-\cos 2y) \right] \right]_0^{\pi} =$$

$$= \frac{\pi^2}{4} - \frac{1}{2}\cdot\frac{1}{4} - \left(-\frac{1}{2} \cdot -\frac{1}{4} \right) = \frac{\pi^2}{4} - \frac{1}{8} - \frac{1}{8} = \frac{\pi^2-1}{4}$$

$$E = \left\{ (x,y)\in\mathbb{R}^2 : |xy| \le 1 \right\}$$

~~A~~. $(-2,-\frac{1}{2})$ è punto interno

Ⓑ $(1,-1)$ è punto di accum.

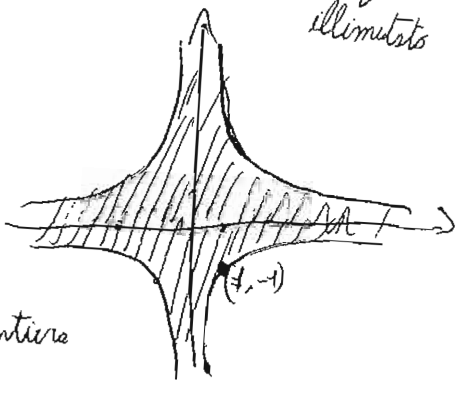
~~C~~. E è aperto → contiene la frontiera

~~D~~. E è limitato → illimitato

$$xy \ge 0 \qquad xy \le 1 \begin{cases} x>0 & x \le \frac{1}{x} \\ x<0 & y \ge \frac{1}{x} \end{cases}$$

$$xy \le 0 \qquad -xy \le 1$$

$$xy \ge -1 \begin{cases} x>0 & y \ge -\frac{1}{x} \\ x<0 & y \le -\frac{1}{x} \end{cases}$$



e $y = \pm\frac{1}{x}$ e $x=-2$, $y=-\frac{1}{2}$ e sta sulla frontiera

$\Rightarrow$ non è interno

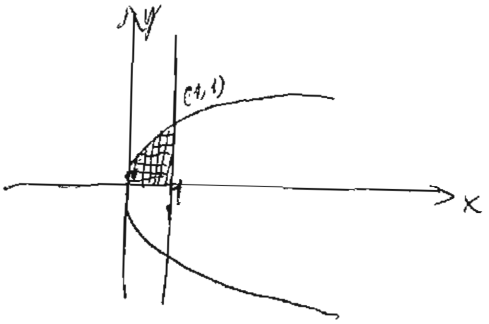
ESERCITAZIONE

28/05/08

①

5.3.12

$y = 0$
$x = 1$
$x = y^2$



BARICENTRO

$$x_G = \frac{1}{m(E)} \int_E x \, dx\, dy$$

$$y_G = \frac{1}{m(E)} \int_E y \, dx\, dy$$

$$m(E) = \int_E dx\,dy = \int_0^1 dx \int_0^{\sqrt{x}} dy = \int_0^1 dx \cdot \sqrt{x}\, d =$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \frac{2}{3}(1-0) = \frac{2}{3} \quad \Longleftrightarrow$$

$A_{\substack{\text{segmento}\\ \text{parabolico}}} = \frac{2}{3} A_{\text{rettangolo}}$

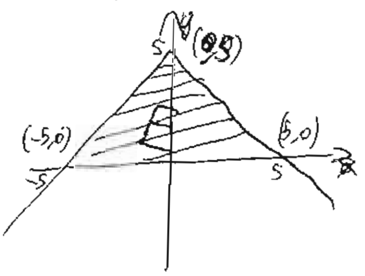
$A_r = 2 \cdot 1 = 2 \quad A_{SP} = \frac{4}{3}$ diviso $\times 2 = \frac{2}{3}$

$\times$ I TRIANGOLI

$$G = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3} \right)$$

B di $E = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 5 - |x|\}$

$x_B = \frac{-5+5+0}{3} \Rightarrow 0 \qquad y_B = \frac{0+0+5}{3} = \frac{5}{3}$



[A] ascissa uguale all'ordinata

[B] $G(0,5)$

☒ ha ordinata $\frac{5}{3}$

[D] ha ascissa positiva

$$x_G = \frac{1}{\frac{2}{3}} \cdot \int_0^1 dx \int_0^{\sqrt{x}} x\, dy = \frac{3}{2} \int_0^1 x \cdot \sqrt{x}\, dx =$$

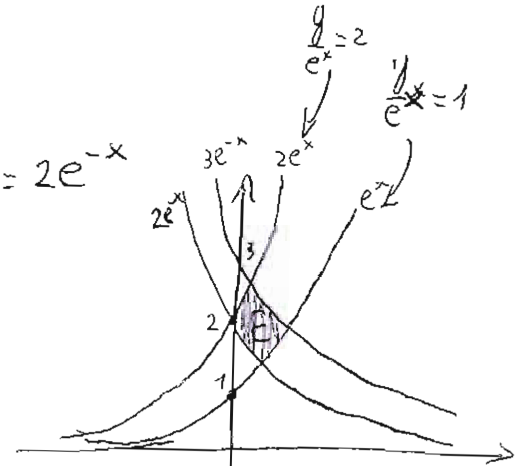
$$= \frac{3}{2} \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^1 = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{5} (1-0) = \frac{3}{5}$$

$$y_G = \frac{1}{\frac{2}{3}} \cdot \int_0^1 dx \int_0^{\sqrt{x}} y\, dy = \frac{3}{2} \int_0^1 dx \cdot \frac{x}{2} =$$

$$= \frac{3}{4} \left[\frac{x^2}{2} \right]_0^1 = \frac{3}{8} \qquad G = \left(\frac{3}{5}, \frac{3}{8} \right)$$

5.3.13.

E è delimitato da $y = e^x$, $y = 2e^x$, $y = 3e^{-x}$ e $y = 2e^{-x}$



$$\int_E f_{\ldots}(x,y)\, dx\, dy \; . \quad \Phi \begin{cases} x = \Phi_1(u,v) \\ y = \Phi_2(u,v) \end{cases} \quad \Phi^{-1} \begin{cases} u = \phi_1^{-1}(x,y) \\ v = \phi_2^{-1}(x,y) \end{cases}$$

$\det J_\phi \qquad \det J_{\phi^{-1}}$

$$\Rightarrow = \int_{\Phi^{-1}(E)} f(\phi(u,v)) \cdot |\det J_\phi|\, du\, dv.$$

$v = \frac{y}{e^x} \quad u = y \cdot e^x$

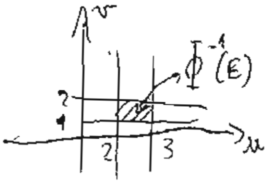
$\searrow 2 \leq \frac{y}{e^x} \leq 3 \qquad 1 \leq \frac{y}{e^x} \leq 2$

$$\Phi^{-1} \begin{cases} v = \frac{y}{e^x} \\ u = y e^x \end{cases} \quad \begin{cases} y = v \cdot e^x \\ u = v \cdot e^x \cdot e^x = v e^{2x} \end{cases} \quad \begin{cases} e^{2x} = \frac{u}{v} \\ \ldots \end{cases} \quad \begin{cases} 2x = \ln \frac{u}{v} \\ \ldots \end{cases} \quad \begin{cases} x = \frac{1}{2} \ln \frac{u}{v} \\ y = v \cdot \sqrt{\frac{u}{v}} = \sqrt{uv} \end{cases}$$

$$\Phi \begin{cases} x = \frac{1}{2} \ln \frac{u}{v} \\ y = \sqrt{uv} \end{cases}$$

$1 \leq v \leq 2$

$2 \leq u \leq 3$



$$J_{\Phi} = \begin{bmatrix} \frac{\partial \phi_1}{\partial u} & \frac{\partial \phi_1}{\partial v} \\ \frac{\partial \phi_2}{\partial u} & \frac{\partial \phi_2}{\partial v} \end{bmatrix} \quad \text{oppure} \quad |\det J_{\Phi}| = \frac{1}{|\det J_{\Phi^{-1}}|} \quad J_{\Phi^{-1}} = \begin{bmatrix} ye^{x} & e^{x} \\ -e^{-x} \cdot y & \frac{1}{e^{x}} \end{bmatrix}$$

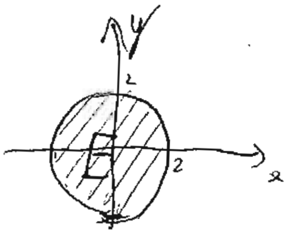
$$\det J_{\Phi^{-1}} = y \cancel{e^{x}} \cdot \frac{1}{\cancel{e^{x}}} - (-\cancel{e^{-x}} \cdot y \cdot \cancel{e^{x}}) = y + y = 2y \qquad |\det J_{\Phi}| = \frac{1}{|2y|} = \frac{1}{2\sqrt{uv}}$$

$$\int_{\Phi^{-1}(E)} 1 \cdot \frac{1}{2\sqrt{uv}}\, du\, dv = \int_2^3 du \int_1^2 dv \frac{1}{2\sqrt{u}\sqrt{v}} = \int_2^3 du \cdot \frac{1}{\sqrt{u}} \left[\sqrt{v}\right]_1^2 = \int_2^3 \frac{\sqrt{2}-1}{\sqrt{u}}\, du =$$

$$= (\sqrt{2}-1) \cdot 2 \int_2^3 \frac{1}{2\sqrt{u}}\, du = (2\sqrt{2}-2)\left[\sqrt{u}\right]_2^3 = (2\sqrt{2}-2)(\sqrt{3}-\sqrt{2}) = 2\sqrt{6} - 4 - 2\sqrt{3} - 2\sqrt{2}.$$

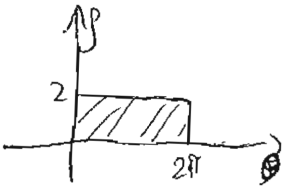
5.3.15

$$\int_E \frac{1}{1+x^2+y^2}\, dx\, dy \qquad E = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$$



Con cerchi, circonferenze,
⇒ coordinate polar
Con x^2+y^2 nella funzion
⇒ coordinate polari

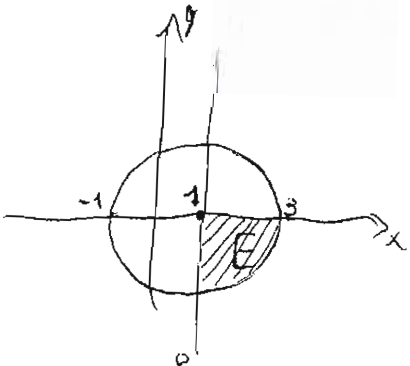
$$\Phi \begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases} \qquad \det J_{\Phi} = \rho \qquad \begin{matrix} \theta \in [0, 2\pi] \\ \rho \in [0, 2] \end{matrix}$$



$$\int_{\Phi^{-1}(E)} \frac{1}{1+\rho^2} \cdot \rho\, d\rho\, d\theta = \int_0^{2\pi} d\theta \frac{1}{2} \int_0^2 \frac{2\rho}{1+\rho^2}\, d\rho = \frac{1}{2} \int_0^{2\pi} d\theta \left[\log(1+\rho^2)\right]_0^2 = \frac{1}{2} \int_0^{2\pi} \log 5\, d\theta = \frac{\log 5}{\cancel{2}} \cdot \cancel{2}\pi = \pi \log 5.$$

$$\int_F xy\, dx\, dy \qquad E = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 4,\ x \geq 1,\ y \leq 0\}$$

$$\Phi^{-1} \begin{cases} x = 1 + \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases} \qquad \det J_{\Phi} = \rho \qquad \begin{matrix} \theta \in [-\frac{\pi}{2}, 0] \\ \rho \in [0, 2] \end{matrix}$$



$$\int_{-\frac{\pi}{2}}^{0} d\theta \int_0^2 (1+\rho\cos\theta)\rho\,\text{sen}\,\theta \cdot \rho\, d\rho = \int_{-\frac{\pi}{2}}^{0} d\theta \int_0^2 d\rho \left(\rho^2 \text{sen}\,\theta + \rho^3 \text{sen}\,\theta\cos\theta\right) =$$

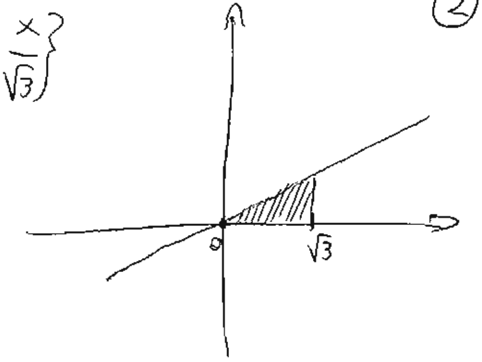
$$= \int_{-\frac{\pi}{2}}^{0} d\theta \left| \frac{\rho^3}{3} \text{sen}\,\theta + \frac{\rho^4}{4} \text{sen}\,\theta\cos\theta \right|_0^2 = \int_{-\frac{\pi}{2}}^{0} d\theta \cdot \left(\frac{8}{3} \text{sen}\,\theta + 4\,\text{sen}\,\theta\cos\theta\right) =$$

$$= \left[-\frac{8}{3}\cos\theta + 2\,\text{sen}^2\theta\right]_{-\frac{\pi}{2}}^{0} = -\frac{8}{3} + 0 - (0 + 2) = -\frac{14}{3}$$

28/03/08 ②

5.3.16.

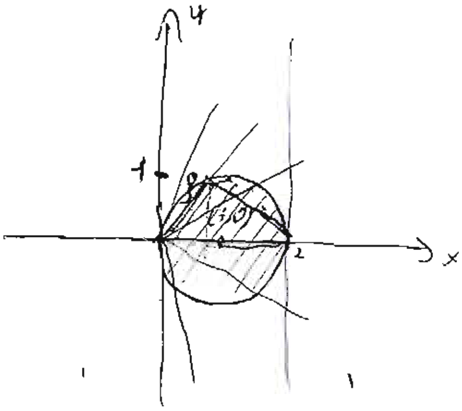
$\int_0^{\sqrt{3}} dx \int_0^{x/\sqrt{3}} dy$ cambio le coordinate $E = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le \sqrt{3}, 0 \le y \le \frac{x}{\sqrt{3}}\right\}$



$0 \le \theta \le \frac{\pi}{6}$ $\qquad \overset{\rho}{\sqrt{3}} = \rho\cos\theta \Rightarrow \rho = \frac{\sqrt{3}}{\cos\theta}$

$\int_0^{\frac{\pi}{6}} d\theta \int_0^{\frac{\sqrt{3}}{\cos\theta}} d\rho \cdot \overset{J_\Phi}{\rho}$.

$\int_0^2 dx \int_{-\sqrt{2x-x^2}}^{+\sqrt{2x-x^2}} dy$ $\qquad E = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2, \ y^2 \le 2x - x^2\right\}$

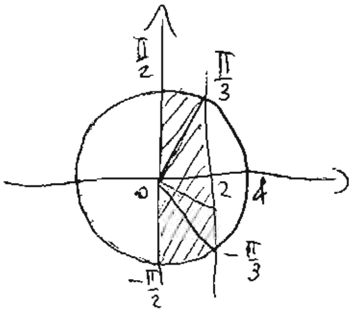


$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$: $\qquad 0 \le \rho \le 2\cos\theta$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} d\rho \cdot \rho$.

$\int_0^2 dx \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dy$ $\qquad E = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2, \ x^2 + y^2 \le 16\right\}$

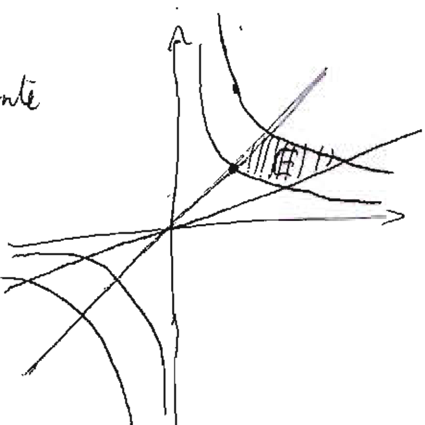
Divido in 3 parti.



- $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{3}$ $\qquad 0 \le \rho \le 4$
- $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ $\qquad 0 \le \rho \le \frac{2}{\cos\theta}$
- $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ $\qquad 0 \le \rho \le 4$

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} d\theta \int_0^4 d\rho \cdot \rho + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^{\frac{2}{\cos\theta}} d\rho \cdot \rho + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^4 d\rho \cdot \rho .$$

$E = \left\{(x,y) \in \mathbb{R}^2 : \frac{4}{x} \le y \le \frac{9}{x}, \ y \le x \le 4y\right\}$ Considerare E nel 1° quadrante

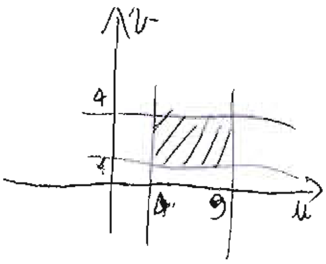


$y = x \quad \frac{x}{y} = 1$

$y = \frac{1}{4}x \quad \frac{x}{y} = 4$ $\Rightarrow 1 \le \frac{x}{y} \le 4$

$xy = 4$
$xy = 9$ $\Rightarrow 4 \le xy \le 9$

$\Phi^{-1}\begin{cases} u = xy \\ v = \frac{x}{y} \end{cases}$ $\qquad \Phi = \begin{cases} x = \frac{u}{y} \\ v = \frac{u}{y} \cdot \frac{1}{y} \end{cases}$ $\qquad \begin{cases} x = \frac{u}{y} \\ y = \underset{1^\circ \text{ quadrante}}{+}\sqrt{\frac{u}{v}} \end{cases}$ $\qquad \begin{cases} x = \sqrt{uv} \\ y = \sqrt{\frac{u}{v}} \end{cases}$



$J_{\Phi^{-1}} \begin{bmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{bmatrix}$ $\quad |\det J_{\Phi^{-1}}| = \left|-\frac{x}{y} - \frac{x}{y}\right| = \left|-\frac{2x}{y}\right| = \left|-\frac{2\sqrt{uv}}{\sqrt{\frac{u}{v}}}\right|$ $\quad |\det J_\Phi| = \frac{1}{2v}$

$$\int_{\phi^{-1}(E)} f(\phi(u,v)) \cdot |\det J_\phi| \, du\,dv = \int_1^4 dv \int_4^9 du \cdot 1 \cdot \frac{1}{2v} = \int_1^4 \frac{1}{2v}(9-4)\,dv = \frac{5}{2}\int_1^4 \frac{1}{v}\,dv = \frac{5}{2}\Big[\log v\Big]_1^4 =$$

$$= \frac{5}{2} \cdot \log 4 = \frac{5}{2} \log 2^2 = \frac{5}{\cancel{2}} \cdot \cancel{2} \log 2 = 5 \log 2 \qquad \underline{AREA} = m(E)$$

$$X_G = \frac{1}{m(E)} \int_E x\,dx\,dy = \frac{1}{5\ln 2}\int_4^9 du \int_1^4 dv \cdot \sqrt{uv} \cdot \frac{1}{2v} = \frac{1}{5\ln 2}\int_4^9 du\,\sqrt{u}\int_1^4 \frac{1}{2\sqrt{v}}\,dv = \frac{1}{5\ln 2}\int_4^9 \sqrt{u}\Big[\sqrt{v}\Big]_1^4 du =$$

$$= \frac{1}{5\ln 2}\int_4^9 \sqrt{u}(2-1)\,du = \frac{1}{5\ln 2}\int_4^9 \sqrt{u}\,du = \frac{1}{5\ln 2}\left[\frac{u^{3/2}}{3/2}\right]_4^9 = \frac{1}{5\ln 2}\left[\frac{2}{3}(27-8)\right] = \frac{1}{5\ln 2} \cdot \frac{38}{3}$$

QUIZ

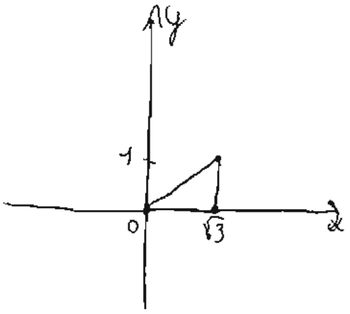
Dato il triangolo T di vertici $(0,0)$ $(\sqrt{3},0)$ $(\sqrt{3},1)$ si consideri $I = \int_T xy\,dx\,dy$, tra le seguenti identità qual è FALSA:

[A] ■ $I = \int_0^1 dy \int_{\sqrt{3}y}^{\sqrt{3}} xy\,dx$

[B] $I = \int_0^{\sqrt{3}} x\,dx \int_0^{\frac{x}{\sqrt{3}}} y\,dy$

[C] $I = \int_0^{\frac{\pi}{6}} d\theta \int_0^{\frac{\sqrt{3}}{\cos\theta}} \rho^3 \operatorname{sen}\theta\cos\theta\,d\rho$

☒ $I = \int_0^{\frac{\pi}{3}} d\theta \int_{\sqrt{3}\cos\theta}^{2} \rho^3 \operatorname{sen}\theta\cos\theta\,d\rho$

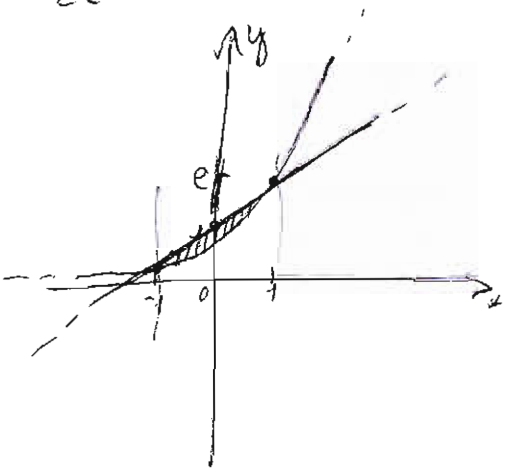


ESERCITAZIONE

29/05/08

① $\int_{\Omega} f(x,y)\,dx\,dy \qquad f(x,y)=x \qquad \Omega$ delimitato da $y=e^x$

$$(*)\; y=\frac{(e^2-1)(x-1)}{2e}+e$$



$$(*)\; \frac{e^2x-x-e^2+1+2e^2}{2e}=\frac{e^2x-x+e^2+1}{2e}\;\cancel{=}\;[illegible]$$

[illegible]
$$\frac{x(e^2-1)+e^2+1}{2e}$$

DOMINIO NORMALE
RISPETTO A X

$$\int_{-1}^{1}dx\int_{e^x}^{\frac{(e^2-1)(x-1)}{2e}+e}x\,dy=\int_{-1}^{1}dx\cdot x\cdot\left(\frac{(e^2-1)(x-1)}{2e}+e-e^x\right)=\int_{-1}^{1}\left(\frac{(x^2-x)(e^2-1)}{2e}+ex-\underset{g\;f'}{xe^x}\right)dx=$$

$$=\left[\frac{e^2-1}{2e}\left(\frac{x^3}{3}-\frac{x^2}{2}\right)+e\frac{x^2}{2}-\left(xe^x-\int e^x dx\right)\right]_{-1}^{1}=\frac{e^2-1}{2e}\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{e}{2}\cancel{-e}\cancel{+e}-\left[\frac{e^2-1}{2e}\left(-\frac{1}{3}-\frac{1}{2}\right)\right.$$

$$\left.+\frac{e}{2}+e^{-1}+e^{-1}\right]=\frac{e^2-1}{2e}\left(\frac{2-3}{6}\right)+\cancel{\frac{e}{2}}-\frac{e^2-1}{2e}\left(\frac{-2-3}{6}\right)-\cancel{\frac{e}{2}}-2e^{-1}=$$

$$=\frac{e^2-1}{2e}\left(-\frac{1}{6}+\frac{5}{6}\right)-\frac{2}{e}=\frac{e^2-1}{3e}-\frac{2}{e}=\frac{e^2-1-6}{3e}=\frac{e^2-7}{3e}$$

② $A=\{(x,y)\in\mathbb{R}^2:\ -1\le x\le 1,\ -1\le y\le 1\}$

$B=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 1,\ x\ge 0\}$

BARICENTRO DI $\Omega=A\setminus B$

$$x_\Omega=\frac{m(A)\,x_A-m(B)\,x_B}{m(A)-m(B)}$$

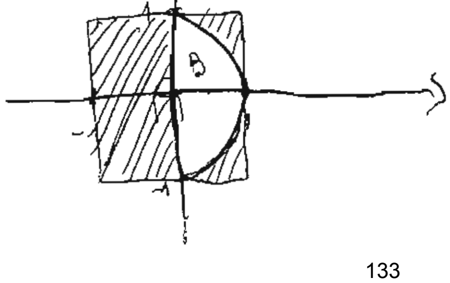
sia (x_Ω,y_Ω) il baricentro geometrico di $\Omega=A\setminus B$

$m(A)=2^2=4$

$m(B)=\frac{\pi 1^2}{2}=\frac{\pi}{2}$

(i) $(x_\Omega,y_\Omega)=\left(\frac{2}{3\pi},0\right)$ (iii) $(x_\Omega,y_\Omega)\in A\cap B$

[✓ (ii)] $(x_\Omega,y_\Omega)=\left(\frac{-4}{3(8-\pi)},0\right)$ (iv) nessuna delle altre



$/\!/\!/$ A\B

$$x_A = \frac{1}{m(A)} \cdot \int_A x\,dx\,dy = \frac{1}{4} \cdot \int_{-1}^{1} dx \int_{-1}^{1} x\,dy = \frac{1}{4}\int_{-1}^{1} dx \cdot x \cdot 2 = \frac{2}{4}\left[\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

$$y_A = \frac{1}{m(A)} \int_A y\,dx\,dy = \frac{1}{4}\int_{-1}^{1} dx \int_{-1}^{1} y\,dy = \frac{1}{4}\int_{-1}^{1} dx\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

$x^2+y^2=1$ $\cancel{x}=\sqrt{1-y^2}$

~~$x_B = \frac{1}{m(B)} \int_B x\,dx\,dy = \frac{1}{\frac{\pi}{2}} \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\,dx = \frac{2}{\pi}\int_{-1}^{1} dy \cdot \left(\frac{1-y^2}{2} - \frac{1-y^2}{2}\right) = 0$~~

~~$y_B = \frac{1}{\frac{\pi}{2}} \int_{-1}^{1} dy \int$~~ $\phi_{AB}^{-1}\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \end{cases}$ $\rho^2 = 1$ $\rho = \pm 1$ $\det J = \rho$ $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\rho \in [0,1]$

~~MBA~~ $x_B \int_\phi \rho\cos\theta\, d\rho\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^1 \rho\cos\theta \cdot \rho \cdot d\rho = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \left[\frac{\rho^3}{3}\right]_0^1 = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta = \frac{1}{3}\left(\text{sen}\frac{\pi}{2} - \text{sen}\left(-\frac{\pi}{2}\right)\right)$

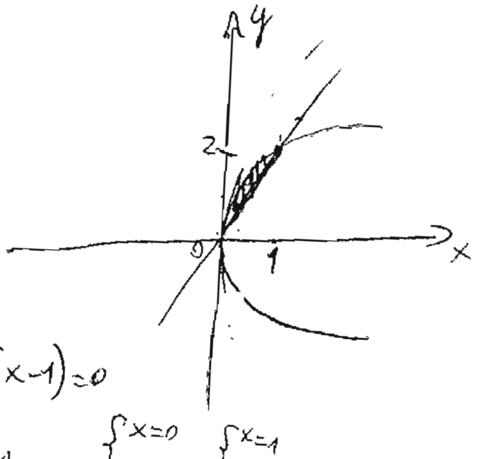
$= \frac{1}{3} \cdot (1+1) = \frac{2}{3}$ $y_B = 0$

$$x_\Omega = \frac{\cancel{4}\,0 - \frac{\pi}{2}\cdot\frac{2}{3}}{4 - \frac{\pi}{2}} = \frac{-\frac{\pi}{3}}{\frac{8-\pi}{2}} = -\frac{\pi}{3}\cdot\frac{2}{8-\pi} = \frac{-2\pi}{3(8-\pi)}$$

$$y_\Omega = \frac{4\cdot 0 - \frac{\pi}{2}\cdot 0}{4-\frac{\pi}{2}} = 0$$

③

$E = \left\{(x,y) \in \mathbb{R}^2 : \frac{y^2}{4} \le x \le \frac{y}{2}\right\}$ l'integrale $\int_E (y-y^3)\,dx\,dy$



(i) $=0$ ~~(iii)~~ $= -\frac{1}{5}$

(ii) $= \frac{2}{5}$ (iv) ∄ poichè E è illimitato

$\begin{cases} \frac{y^2}{4} \le x \\ x \le \frac{y}{2} \end{cases}$ $\begin{cases} x \geq \frac{1}{4}y^2 \\ y \geq 2x \end{cases}$ $\begin{cases} x = \frac{1}{4}y^2 \\ y = 2x \end{cases}$ $\begin{cases} y = 2x \\ x = \frac{4x^2}{4} \end{cases}$ $x(x-1)=0$

$\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} x=1 \\ y=2 \end{cases}$

$$\int_0^1 dx \int_{2x}^{\sqrt{\cancel{4x}}} (y-y^3)\,dy = \int_0^1 dx \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_{2x}^{\sqrt{\cancel{4x}}} = \int_0^1 dx\left(\frac{\overset{2}{\cancel{4}}x}{\cancel{2}} - \frac{\overset{4}{\cancel{16}}x^2}{\cancel{4}} - 2x^2 + 4x^4\right) =$$

$$\int_0^1 (4x^4 - 6x^2 + 2x)\,dx = \left[\frac{x^5}{5}\cdot 4 - 2x^3 + x^2\right]_0^1 = \frac{4}{5} - 2 + 1 = \boxed{-\frac{1}{5}}$$ [illegible]

[illegible]

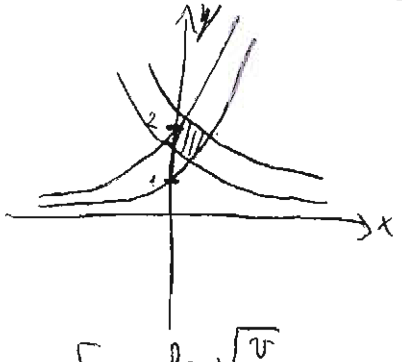
29/05/08

$\boxed{2}$

⑨ $\displaystyle\int_\Omega \frac{2y^2}{e^x}\cos(ye^x)\,dx\,dy$ dove $\Omega=\{(x,y)\in\mathbb{R}^2 : e^x\le y\le 2e^x,\ \frac{\pi}{2}\le ye^x\le\pi\}$

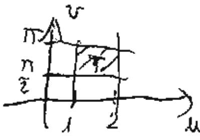
$$y=e^x \qquad 1\le \underbrace{\frac{y}{e^x}}_{u}\le 2 \qquad ye^x\ge\frac{\pi}{2} \qquad \frac{\pi}{2}\le \underbrace{ye^x}_{v}\le\pi$$

$$y=2e^x \qquad\qquad ye^x\le\pi$$



$$\Phi^{-1}\begin{cases}u=\frac{y}{e^x}\\ v=ye^x\end{cases} \qquad \Phi\begin{cases}y=e^x u\\ v=e^x\cdot u\cdot e^x\end{cases} \qquad \begin{cases}y=e^x\cdot u\\ e^{2x}=\frac{v}{u}\end{cases} \qquad \begin{cases}e^x=\sqrt{\frac{v}{u}}\\ y=\sqrt{\frac{v}{u}}\cdot u\end{cases}$$

$$\begin{cases}x=\ln\sqrt{\frac{v}{u}}\\ y=u\cdot\sqrt{\frac{v}{u}}\end{cases}$$



$$J^{-1}_{(u,v)}\begin{vmatrix}-e^{-x}\cdot y & \frac{1}{e^x}\\ ye^x & e^x\end{vmatrix} \qquad |\det J^{-1}_{(u,v)}|=|-y-y|=|-2y|=2y$$

$$\det J_{(u,v)}=\frac{1}{2u\sqrt{\frac{v}{u}}}$$

$$\int_E\frac{2y^2}{e^x}\cos(ye^x)\,dx\,dy=\int_T\frac{2u^2\cdot\frac{v}{u}}{e^{\ln\sqrt{\frac{v}{u}}}}\cdot\cos\left(u\cdot\sqrt{\frac{v}{u}}\cdot e^{\ln\sqrt{\frac{v}{u}}}\right)\cdot\frac{1}{2\cancel{u}\sqrt{\frac{v}{u}}}\,du\,dv=$$

$$=\int_T\frac{\cancel{2u}\cancel{v}}{\sqrt{\frac{v}{u}}}\cdot\cos\left(\cancel{u}\cdot\frac{v}{\cancel{u}}\right)\cdot\frac{1}{\cancel{2u}\sqrt{\frac{v}{u}}}\,du\,dv=\int_T u\cos v\,du\,dv=$$

$$=\int_1^2 du\int_{\frac{\pi}{2}}^{\pi}u\cos v\,dv=\int_1^2 u\,du\,\Big[\operatorname{sen}v\Big]_{\frac{\pi}{2}}^{\pi}=\int_1^2 u\,du\,(0-1)=-\int_1^2 u\,du=-\left[\frac{u^2}{2}\right]_1^2=$$

$$=-\left(\frac{4}{2}-\frac{1}{2}\right)=-\frac{3}{2}$$

④ Dato l'insieme $\Omega=\{(x,y,z)\in\mathbb{R}^3 : 0\le 4z\le x^2+y^2\le 4\}$ calcolate

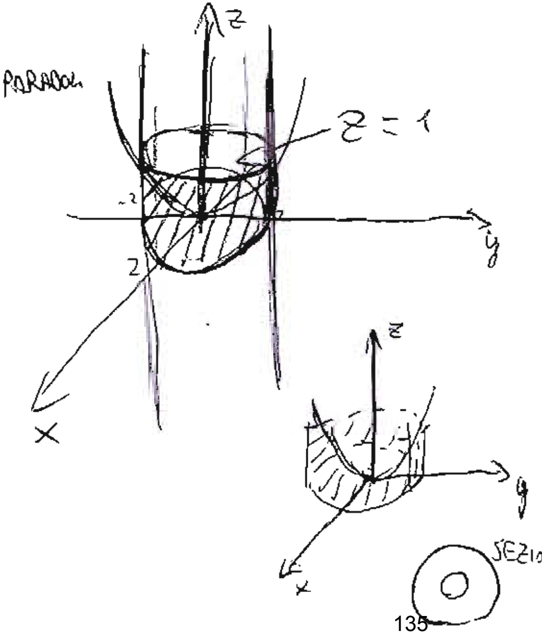
$\displaystyle\int_\Omega(x+y+z)\,dx\,dy\,dz$.

$$\begin{cases}4z\ge 0\\ 4z\le x^2+y^2 \quad z\le\frac{x^2}{4}+\frac{y^2}{4}\text{ PARABOL}\\ x^2+y^2\le 4\end{cases}$$

$$\begin{cases}z=\frac{1}{4}(x^2+y^2)\\ x^2+y^2=4\end{cases} \quad z=1$$

~~CILINDRO~~

~~$\int_{-2}^{2}dx\int_{-2}^{2}dy\int^{1}$~~



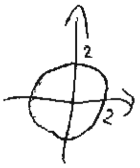
$$\int_\Omega x\,dx\,dy\,dz + \int_\Omega y\,dx\,dy\,dz + \int_\Omega z\,dx\,dy\,dz$$

TECNICA X FILI

$X_G=0$ per simmetria $\quad Y_G=0$ per simmetria $\quad Z_G$

Entro dal cerchio e esco dal paraboloide

$$\int_{x^2+y^2\le4} dx\,dy \int_0^{\frac{1}{4}(x^2+y^2)} z\,dz = \int_{x^2+y^2\le4} dx\,dy \left[\frac{z^2}{2}\right]_0^{\frac{1}{4}(x^2+y^2)} = \int_{x^2+y^2\le4} \frac{1}{32}(x^2+y^2)^2\,dx\,dy = \frac{1}{32}\int_{x^2+y^2\le4}(x^2+y^2)^2\,dx\,dy$$



$$\begin{cases} x=\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta \end{cases} \quad \begin{matrix} \rho\in[0,2] \\ \theta\in[0,2\pi] \end{matrix}$$

$$= \frac{1}{32}\int_0^{2\pi} d\theta \int_0^2 \left[\rho^2\right]^2 \cdot \rho\,d\rho = \frac{1}{32}\int_0^{2\pi} d\theta \left[\frac{\rho^6}{6}\right]_0^2 = \frac{1}{\cancel{32}}\cdot \overset{8}{2}\pi \cdot \frac{\overset{2}{\cancel{64}}}{\cancel{6}3} = \frac{2}{3}\pi.$$

ESERCITAZIONE

4/6/08

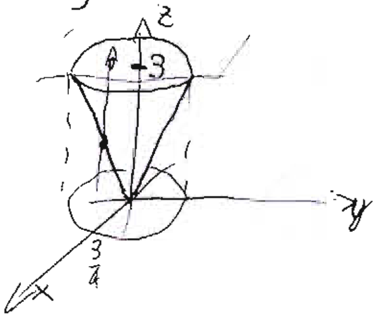
Calcolare il volume di $E=\left\{(x,y,z)\in\mathbb{R}^3 : 4\sqrt{x^2+y^2}\leq z\leq 3\right\}$.

[✓ A.] $\frac{9\pi}{16}$ C. $\frac{8\pi}{27}$

B. $\frac{9\pi}{8}$ D. nessuna

$z=4\sqrt{x^2+y^2}$

$z=3$



$$V_{cono}=\frac{1}{3}A_b\cdot h=\frac{1}{3}\pi r^2\cdot h^{\swarrow 3}$$

$$\begin{cases} z=4\sqrt{x^2+y^2} \\ z=3 \end{cases} \quad 3=4\sqrt{x^2+y^2} \Rightarrow \frac{9}{16}=x^2+y^2 \quad r=\frac{3}{4}$$

$$V_{cono}=\frac{1}{\cancel{3}}\cdot\pi\cdot\frac{9}{16}\cdot\cancel{3}=\frac{9\pi}{16}$$

oppure

$$m(E)=\int_E 1\,dx\,dy\,dz=\int_{x^2+y^2\leq\frac{9}{16}}dx\,dy\int_{4\sqrt{x^2+y^2}}^{3}dz \quad \dots$$

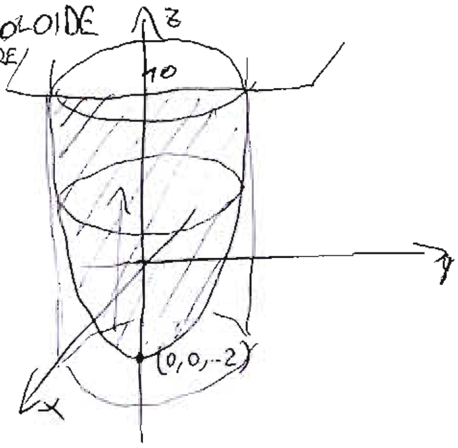
· Calcolare il volume di $E=\left\{(x,y,z)\in\mathbb{R}^3 : 3x^2+3y^2-2\leq z\leq 10\right\}$

A. 16π C. $\frac{76\pi}{3}$

B. $\frac{64\pi}{3}$ [✓ D.] 24π

$z=3x^2+3y^2-2$ = PARABOLOIDE CIRCOLARE

$z=10$



$$\begin{cases} 3x^2+3y^2-2=z \\ z=10 \end{cases}$$

$$V_{PAR.}=\frac{1}{2}V_{\substack{CILINDRO\\CIRCOSCRITTO}}$$

$V_{CIL}=\pi R^2\cdot h \quad h=12$

$3(x^2+y^2)=12 \quad x^2+y^2=4 \quad R=2 \quad V_{CIL}=\pi\cdot 4\cdot 12=48\pi$

$$V_{PAR}=\frac{1}{2}\cdot 48\pi=24\pi$$

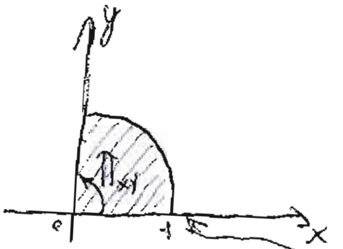
oppure

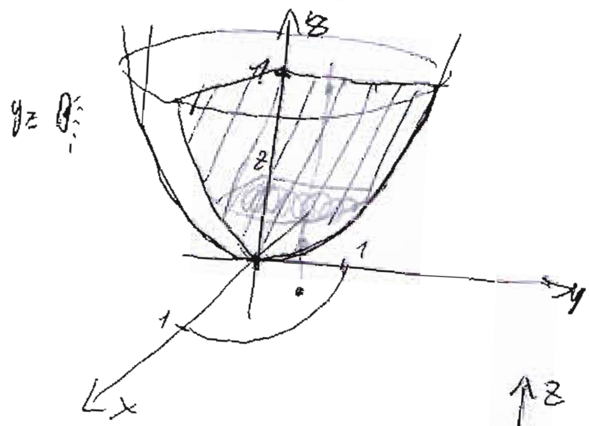
$$\int_{x^2+y^2\leq 4}dx\,dy\int_{3x^2+3y^2\leq z}^{10}dz$$

↑
PROIEZIONE SUL PIANO xy

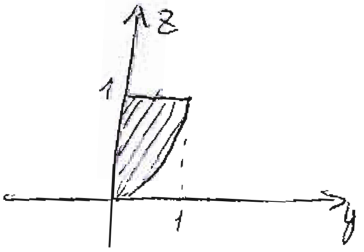
VOLUME $E = \{(x,y,z) \in \mathbb{R}^3 : 0 < x, 0 < y, x^2+y^2 \le z \le 1\}$ $\int_E dx\,dy\,dz$

$z = x^2+y^2$ PARABOLOIDE





$$\begin{cases} z = x^2+y^2 \\ z = 1 \end{cases} \quad x^2+y^2=1$$



FILI

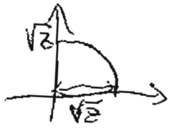
$$\int_{\Pi_{xy}} dx\,dy \int_{x^2+y^2}^{1} dz = \int_{\Pi_{xy}} dx\,dy \left(1-x^2-y^2\right) = \int_{\begin{cases} x^2+y^2\le 1 \\ x\ge 0 \\ y \ge 0\end{cases}} \left(1-x^2-y^2\right) dx\,dy =$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 d\rho \left(1-\overset{\overset{x^2+y^2}{\downarrow}}{\rho^2}\right)\cdot \overset{\overset{J}{\downarrow}}{\rho} = \int_0^{\frac{\pi}{2}} d\theta \left| \frac{\rho^2}{2} - \frac{\rho^4}{4} \right|_0^1 = \frac{\pi}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{8}$$

STRATI → sezioni con piani // xy

Trovo quarti di cerchio con raggio variabili

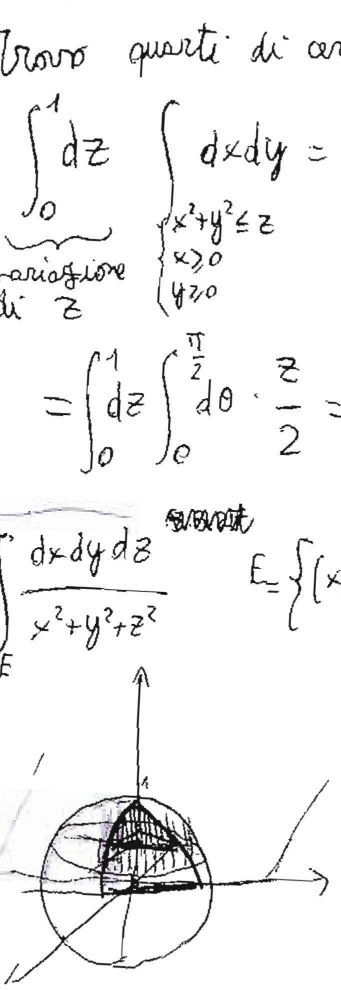
$$\begin{cases} z = x^2+y^2 \\ z = K \end{cases} \quad x^2+y^2 = K \quad R = \sqrt{K} = \sqrt{z}$$

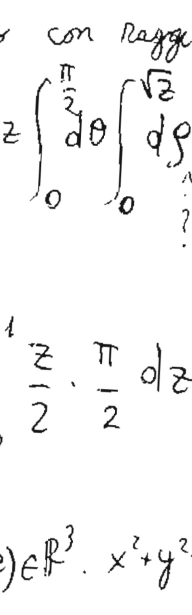


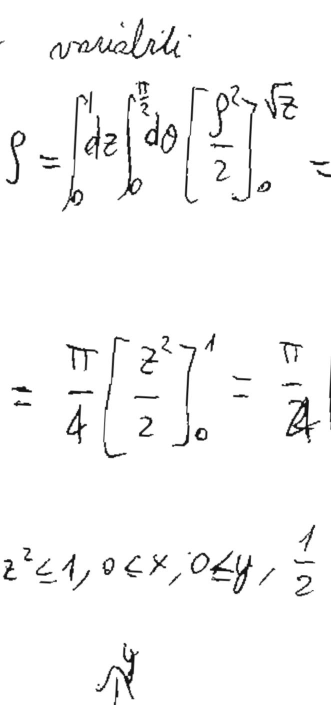
$$\underbrace{\int_0^1 dz}_{\text{variazione di } z} \int_{\begin{cases} x^2+y^2\le z \\ x\ge 0 \\ y\ge 0\end{cases}} dx\,dy = \int_0^1 dz \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sqrt{z}} d\rho\cdot\underset{?}{\rho} = \int_0^1 dz \int_0^{\frac{\pi}{2}} d\theta \left[\frac{\rho^2}{2}\right]_0^{\sqrt{z}} =$$

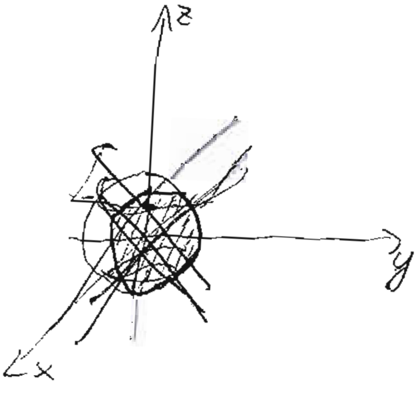
$$= \int_0^1 dz \int_0^{\frac{\pi}{2}} d\theta \cdot \frac{z}{2} = \int_0^1 \frac{z}{2}\cdot\frac{\pi}{2}\,dz = \frac{\pi}{4}\left[\frac{z^2}{2}\right]_0^1 = \frac{\pi}{4}\left(\frac{1}{2}\right) = \frac{\pi}{8}$$

$$\int_E \frac{dx\,dy\,dz}{x^2+y^2+z^2} \quad E = \left\{(x,y,z)\in\mathbb{R}^3 .\ x^2+y^2+z^2 \le 1, 0 \le x, 0 \le y, \frac{1}{2} \le z\right\}$$









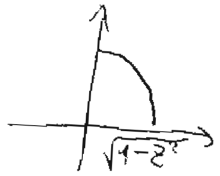
$$\begin{cases} x^2+y^2+z^2=1 \\ z = \frac{1}{2} \end{cases} \quad x^2+y^2 = \frac{3}{4}$$

$$\int_{\substack{x^2+y^2\le\frac{3}{4}\\ x\geq 0\\ y\geq 0}} dx\,dy \int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\,dz \quad \dots$$

STRATI

$$\int_{\frac{1}{2}}^{1} dz \int_{\substack{x^2+y^2\le 1-z^2\\ x\geq 0\\ y\geq 0}} \frac{dx\,dy}{x^2+y^2+z^2} =$$

$$\begin{cases} x^2+y^2+z^2=1 \\ z=k \end{cases} \quad x^2+y^2=1-k^2 \quad r=\sqrt{1-k^2}=\sqrt{1-z^2}$$



$$= \int_{\frac{1}{2}}^{1} dz \int_0^{\frac{\pi}{2}} d\theta \frac{1}{2} \int_0^{\sqrt{1-z^2}} d\rho \cdot \rho \cdot \frac{1 \cdot 2}{\rho^2+z^2} = \int_{\frac{1}{2}}^{1} dz \int_0^{\frac{\pi}{2}} d\theta \cdot \frac{1}{2} \left[\ln(\rho^2+z^2)\right]_0^{\sqrt{1-z^2}} =$$

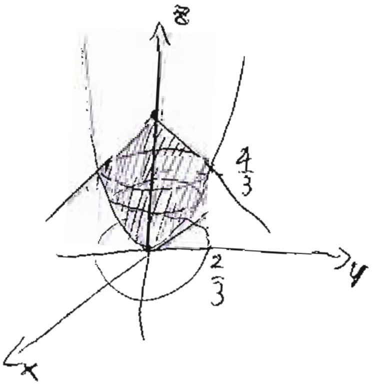
$$= \frac{1}{2}\int_{\frac{1}{2}}^{1} dz \cdot \frac{\pi}{2} \cdot \left(\overset{0}{\cancel{\ln(1-z^2+z^2)}} - \ln z^2\right) = -\frac{\pi}{4}\int_{\frac{1}{2}}^{1} 2\ln z\,dz = -\frac{\pi}{2}\int_{\frac{1}{2}}^{1} \ln z\,dz =$$

$$= -\frac{\pi}{2}\left(z\log z - z\right)_{\frac{1}{2}}^{1} = -\frac{\pi}{2}\left(\cancel{1\cdot\log 1} - 1 - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\right) = -\frac{\pi}{2}\left(-\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right) =$$

$$= \frac{\pi}{4}\left(1 + \log\frac{1}{2}\right)$$

S.3.27

$E = \left\{(x,y,z) \in \mathbb{R}^3 : 3x^2+3y^2 \le z \le 2-\sqrt{x^2+y^2}\right\}$ Volume.



$\begin{cases} z = 3(x^2+y^2) & \text{PARABOLOIDE} \\ z = 2 - \sqrt{x^2+y^2} & \end{cases}$ ~~$(z-2)\ 2-\sqrt{x^2+y^2}\ \ x^2+y^2+(z-2)^2=0$~~

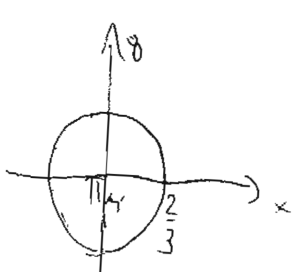
CONO CONC. BASSO TRASLATO DI 2

$3x^2+3y^2 = 2-\sqrt{x^2+y^2} \qquad \sqrt{x^2+y^2} = t$

$3t^2 = 2-t \qquad 3t^2+t-2=0 \qquad t = \dfrac{-1\pm\sqrt{1+24}}{6} = \dfrac{-1\pm 5}{6} = \begin{cases} -1 \\ \frac{2}{3} \end{cases}$

$t=-1$ IMP. $\quad t=\frac{2}{3} \quad \sqrt{x^2+y^2}=\frac{2}{3} \quad x^2+y^2=\frac{4}{9} \quad r=\frac{2}{3}$

$= 3\cdot\frac{4}{9} = \frac{4}{3}$ quota di intersezione



F(L)

$$\int_{x^2+y^2\le\frac{4}{9}} dx\,dy \int_{3x^2+3y^2}^{2-\sqrt{x^2+y^2}} dz = \int_{x^2+y^2\le\frac{4}{9}} dx\,dy \cdot \left(2-\sqrt{x^2+y^2}-3(x^2+y^2)\right) = \int_0^{2\pi} d\theta \int_0^{\frac{2}{3}} d\rho\cdot\rho\cdot\left(2-\rho-3\rho^2\right) =$$

$$= \int_0^{2\pi} d\theta \int_e^{\frac{2}{3}} \left(2\rho-\rho^2-3\rho^3\right) d\rho = 2\pi\cdot\left(\rho^2-\frac{\rho^3}{3}-3\frac{\rho^4}{4}\right)_0^{\frac{2}{3}} = 2\pi\left(\frac{4}{9}-\frac{1}{3}\cdot\frac{8}{27}-\cancel{3}\cdot\frac{1}{\cancel{4}}\cdot\frac{\cancel{16}^{4}}{81_{27}}\right) =$$

$$= 2\pi\left(\frac{36-8-12}{81}\right) = 2\pi\cdot\frac{16}{81} = \frac{32}{81}\pi$$

VIA ELEMENTARE

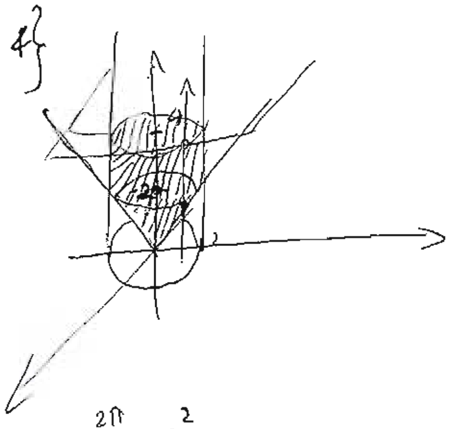
$$V_{PAR} = \frac{1}{2}V_{CIL} = \frac{1}{2}\cdot\pi\cdot R^2\cdot h = \frac{1}{\cancel{2}}\cdot\pi\cdot\frac{\cancel{4}^{2}}{9}\cdot\frac{4}{3} = \frac{8}{27}\pi$$

$$V_{CONO} = \frac{1}{3}\cdot\pi R^2\cdot h = \frac{1}{3}\cdot\frac{4}{9}\cdot\left(2-\frac{4}{3}\right)\pi = \pi\frac{4}{27}\left(\frac{2}{3}\right) = \frac{8}{81}\pi$$

$$V = V_{PAR}+V_{CONO} = \frac{8}{27}\pi+\frac{8}{81}\pi = \frac{24+8}{81}\pi = \frac{32}{81}\pi$$

S.3.28

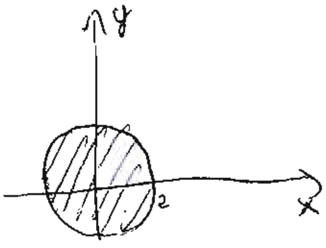
$$\int_E 2z\,dx\,dy\,dz \qquad E=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le 4,\ \sqrt{x^2+y^2}\le z\le 4\right\}$$



$x^2+y^2\le 4$ ~~PARABOLOIDE~~ CILINDRO

$z=\sqrt{x^2+y^2}$ CONO

$$\begin{cases} x^2+y^2\le 4 \\ z=\sqrt{x^2+y^2} \end{cases} \qquad z=2$$



$$\int_{x^2+y^2\le 4} dx\,dy \int_{\sqrt{x^2+y^2}}^{4} 2z\,dz = \int_{x^2+y^2\le 4} dx\,dy \left|z^2\right|_{\sqrt{x^2+y^2}}^{4} = \int_{x^2+y^2\le 4} 16-(x^2+y^2)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^2 d\rho\,(16-\rho^2)\cdot\rho =$$

$$= \int_0^{2\pi} d\theta \int_0^2 (16\rho-\rho^3)\,d\rho = 2\pi\left(8\rho^2-\frac{\rho^4}{4}\right)_0^2 = 2\pi(32-4) = 56\pi$$

11.06.07

(2) $\varphi : [0,6] \to \mathbb{R}^2 \qquad \varphi(t) = (x(t), y(t))$

I) $\begin{cases} x(t) = t \\ y(t) = t^3 \end{cases} \quad t \in [0,1] \quad P_I(0,0) \quad P_F(1,1)$

II) $\begin{cases} x(t) = \cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \\ y(t) = 1 + 2\,\text{sen}\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \end{cases} \quad t \in [1,2] \quad P_I(1,1) \quad P_F(0,3)$

III) $\begin{cases} x(t) = -3t + 6 \\ y(t) = 9 - 3t \end{cases} \quad t \in [2,3] \quad P_I(0,3) \quad P_F(-3,0)$

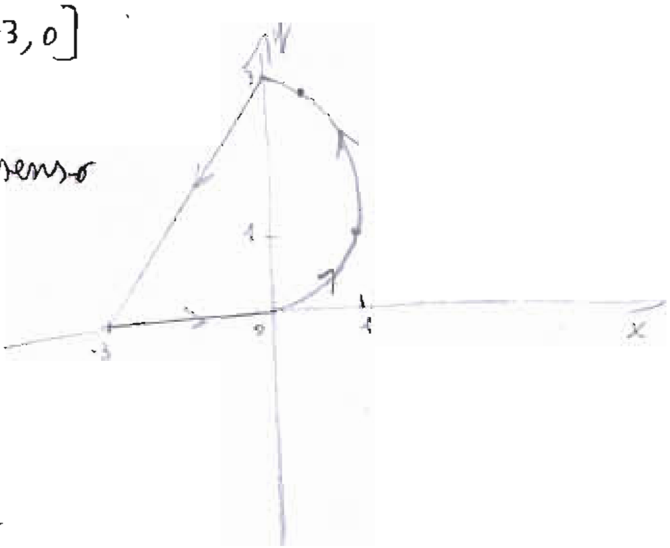
IV) $\begin{cases} x(t) = t - 6 \\ y(t) = 0 \end{cases} \quad t \in]3,6] \quad P_I(-3,0) \quad P_F(0,0)$

I) $y = x^3 \quad x \in [0,1]$

II) $x^2 + \left(\frac{y-1}{2}\right)^2 = \cos^2\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) + 4\,\text{sen}^2\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \qquad x^2 + \frac{(y-1)^2}{4} = 1 \quad x \in [0,1]$

III) $y = x + 3 \quad x \in [-3,0]$ IV) $y = 0 \quad x \in [-3,0]$

La curva percorre il sostegno in senso antiorario



III) $P\left(\frac{1}{2}, 1+\sqrt{3}\right)$

$\begin{cases} \frac{1}{2} = \cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \\ \cancel{1}+\sqrt{3} = \cancel{1} + 2\,\text{sen}\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \end{cases} \qquad \begin{cases} \cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) = \frac{1}{2} \\ \text{sen}\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2} \end{cases}$

$\varphi' : \begin{cases} x'(t) = -\text{sen}\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \cdot \frac{\pi}{2} \\ y'(t) = 2\cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \cdot \frac{\pi}{2} \end{cases} \qquad \varphi'(P) = \begin{cases} x'(t) = -\frac{\sqrt{3}\pi}{4} \\ y'(t) = \frac{\pi}{2} \end{cases}$

$\underbrace{\varphi'(t) = \begin{cases} x(t) = \frac{1}{2} - \frac{\sqrt{3}\pi}{4} t \\ y(t) = 1 + \sqrt{3} + \frac{\pi}{2} t \end{cases} \quad t \in \mathbb{R}}_{\text{equazione parametrica}}$

$\begin{cases} t = \frac{-4x+2}{\sqrt{3}\pi} \\ y = 1 + \sqrt{3} + \frac{\pi}{2} \cdot \frac{2-4x}{\sqrt{3}\pi} \end{cases}$

$y = \frac{\sqrt{3}+3+1-2x}{\sqrt{3}} =$

$\underbrace{y = \frac{3+4\sqrt{3}-2\sqrt{3}x}{3}}_{\text{equazione cartesiana}}$

(1) $y''-7y'+10y=3e^{2x}$ 1) $y''-7y'+10y=0$ $\lambda^2-7\lambda+10=0$

$(\lambda-2)(\lambda-5)=0$ $\lambda=2$, $\lambda=5$ $y_0(x)=c_1\cdot e^{2x}+c_2e^{5x}$

2) $f(x)=3e^{2x}$ $v(x)=(ke^{2x})x$ $v'(x)=ke^{2x}+2xke^{2x}=ke^{2x}(1+2x)$

$v''(x)=2ke^{2x}(1+2x)+2ke^{2x}=2ke^{2x}(2+2x)=4ke^{2x}(1+x)$

$4k\cancel{e^{2x}}(1+x)-7k\cancel{e^{2x}}(1+2x)+10kx\cancel{e^{2x}}=3\cancel{e^{2x}}$ $y_p(x)=-xe^{2x}$

$k(4+\cancel{4x}-7-\cancel{14x}+\cancel{10x})=3$ $-3k=3$ $k=-1$

$y_g(x)=c_1e^{2x}+c_2e^{5x}-xe^{2x}$

---

(3) $f(x,y)=xy(6-x-y)=$
$=6xy-x^2y-xy^2$

$\frac{\partial f}{\partial x}=6y-2xy-y^2$ $\frac{\partial f}{\partial y}=6x-x^2-2xy$

$$\begin{cases}6y-2xy-y^2=0\\6x-2xy-x^2=0\end{cases}$$

$$\begin{cases}y(6-2x-y)=0\\x(6-2y-x)=0\end{cases}\quad\begin{cases}y=0\\x=0\end{cases}\quad\begin{cases}y=0\\6-2y-x=0\end{cases}\quad\begin{cases}6-2x-y=0\\x=0\end{cases}\quad\begin{cases}6-2x-y=0\\6-2y-x=0\end{cases}$$

$$\begin{cases}y=6-2x\\6-12+4x-x=0\end{cases}\quad\begin{cases}x=2\\y=2\end{cases}$$

$P_0(0,0)$ $P_1(6,0)$ $P_2(0,6)$ $P_3(2,2)$

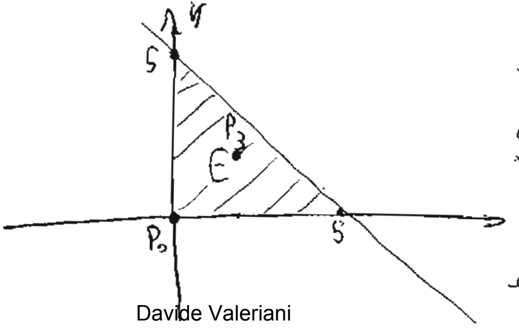
$\frac{\partial^2 f}{\partial x^2}=-2y$ $\frac{\partial^2 f}{\partial x\partial y}=6-2x-2y$ $\frac{\partial^2 f}{\partial y\partial x}=6-2x-2y$ $\frac{\partial^2 f}{\partial y^2}=-2x$

$$H_f(x)=\begin{bmatrix}-2y & 6-2x-2y\\6-2x-2y & -2x\end{bmatrix}\qquad \det H_f(x)=4xy-(6-2x-2y)^2$$

$\det H_f(P_0)=-36<0$ punto di sella $\det H_f(P_1)=-36<0$ punto di sella

$\det H_f(P_2)=-36<0$ punto di sella $\det H_f(P_3)=16-4=12>0$ $\frac{\partial^2 f}{\partial x^2}(P_3)=-4<0$ punto di massim

$$E=\begin{cases}x\geq 0\\y\geq 0\\y\leq -x+5\end{cases}$$



Esistono massimo e minim
assoluti per il teorema d
Weierstrass dato che E
è un insieme chiuso
e limitato

$\begin{cases} x(t)=t \\ y(t)=0 \end{cases}$ $t\in[0,5]$ $f(\varphi(t))=0$ $P=(0,0)$ già trovato

$\begin{cases} x(t)=0 \\ y(t)=t \end{cases}$ $t\in[0,5]$ $f(\varphi(t))=0$ $P=(0,0)$ già trovato

$\begin{cases} x(t)=t \\ y(t)=-t+5 \end{cases}$ $t\in[0,5]$ $f(\varphi(t))=t(-t+5)(6-\cancel{t}+\cancel{t}-5)=-30t(-t+5)=30t^2-150t$

$f'(\varphi(t))=60t-150$ $f'(\varphi(t))=0$ $60t-150=0$ $t=\frac{150}{60}=\frac{5}{2}$ $\begin{cases} x=\frac{5}{2} \\ y=\frac{5}{2} \end{cases}$

$f\left(\frac{5}{2},\frac{5}{2}\right)=\frac{5}{2}\cdot\frac{5}{2}\left(6-\frac{5}{2}-\frac{5}{2}\right)=\frac{25}{4}\left(\frac{12-5-5}{2}\right)=\boxed{\frac{25}{4}}$

$f(0,0)=\boxed{0}$ $f\left(\frac{5}{2},\frac{5}{2}\right)=\boxed{\frac{25}{4}}$ $f(2,2)=\boxed{8}$ $\max_E f(x,y)=8=f(2,2)$

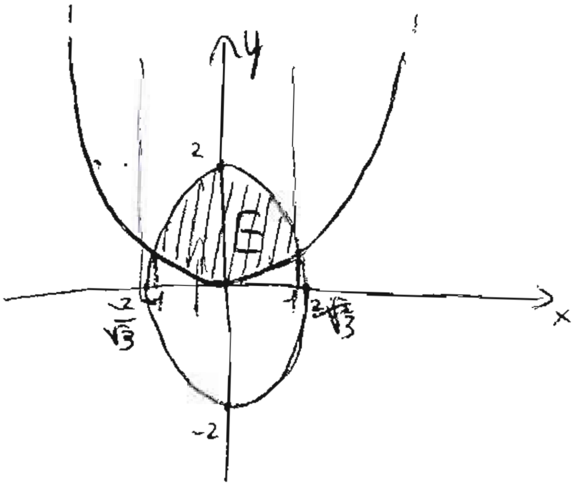
$\min_E f(x,y)=0=f(0,0)$

(4) $\begin{cases} 3x^2+y^2\le 4 \\ y\ge x^2 \end{cases}$ $\begin{cases} \left(\frac{x}{\frac{1}{\sqrt{3}}}\right)^2+\left(\frac{y}{1}\right)^2\le 4 \\ y\ge x^2 \end{cases}$

$\begin{cases} \left(\frac{x}{\frac{2}{\sqrt{3}}}\right)^2+\left(\frac{y}{2}\right)^2\le 1 \\ y\ge x^2 \end{cases}$ $\begin{cases} 3x^2+y^2=4 \\ y=x^2 \end{cases}$ $\begin{cases} 3x^2+x^4=4 \\ y=x^2 \end{cases}$

$t=x^2$ $t^2+3t-4=0$ $(t+4)(t-1)=0$

$t=-4$ $x^2=-4$ MAI $t=1$ $x=\pm 1$



$\int_E x^2y\,dx\,dy$ dominio normale rispetto asse x

$$\int_{-1}^{1}dx\int_{x^2}^{\sqrt{4-3x^2}}x^2y\,dy=\int_{-1}^{1}x^2\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{4-3x^2}}dx=\int_{-1}^{1}x^2\cdot\left[\frac{4-3x^2}{2}-\frac{x^4}{2}\right]dx=$$

$$=\frac{1}{2}\int_{-1}^{1}\left[4x^2-3x^4-x^6\right]dx=\frac{1}{2}\left[4\frac{x^3}{3}-3\frac{x^5}{5}-\frac{x^7}{7}\right]_{-1}^{1}=\frac{1}{2}\left(\frac{4}{3}-\frac{3}{5}-\frac{1}{7}+\frac{4}{3}-\frac{3}{5}-\frac{1}{7}\right)=$$

$$=\frac{1}{2}2\left(\frac{140-63-15}{105}\right)=+\frac{62}{105}\ \checkmark$$